Atomic excitation & energy states

Excitation :- It is the phenomenon of addition of a discrete amount of energy called excitation energy to an atom that results in its alteration ordinarily from the condition of lowest energy (ground state) to one of higher energy (excited state)

In atomic systems, the excited states are not contineously destributed but have only certain discrete energy values. Thus external energy i.e excitation energy can be absorbed only in correspondingly discrete amounts.

Nuclear excitation energies are roughly 1,000,000 times greater than atomic excitation energies.

The excitation energy stored in an excited atom is radiated usually as visible light from atoms as they return to their ground states. This energy can also be lost by collision.

The process of excitation is one of the major means by which matter absorbs pulses of electromagnetic energy (photons), such as light, and by which it is heated or ionised by the impact of charged particles such as electrons or α- particles.

In atoms, the excitation energy is absorbed by the orbiting electrons that are raised to higher distinct energy levels.

The life time of a system in an excited state is usually short, spontaneous and induced emission of a quantum of energy (such as a photon or a phonon) usually occurs shortly after the system is promoted to excited state, Returning the system to a lower energy level is often loosely described as decay and is the inverse of excitation.

Long lived excited states are often called metastable.

Interaction of external energy with atomic energy states

A quantum mechanical system or particle that is bound or confined spatially can take on certain descrete values of energy. These discrete values are called energy levels.

It is commonly used for the energy levels of electrons in atoms which are bound by the electric field of nucleus.

The energy state spectrum of a system with such discrete energy levels is said to be quantized.

If an atom is at the lowest possible energy level it and its electrons are said to be in the ground state. If it is at a higher level, it is said to be excited.

If more than one quantum mechanical state is at the same energy, the energy levels are degenerate. They are then called degenerate energy levels.

Suppose light energy incident on a particle or atom. As per Einstein light is a wave which carries quanta or photons. These are discrete bundles of energy. It carries minimum energy unit of light.

Energy of photon = $E = h\nu$.

Interaction of radiation with atoms is better explained using concept of photon rather than by wave concept.

Energy exchange can take place only at certain discrete values for which the photon energy is the minimum energy unit that light can give or accept.

Light belong to the group of electromagnetic (EM) waves.

Light incident on an atom may under-go absorption depending upon nature of it. Decrease in intensity of light is called attenuation.

When a light beam encounters atoms which have smaller sizes than the wavelength of light i,e $d \ll \lambda$, then it is re-directed into different direction called scattering.

Atoms are distributed in permitted orbits with specific amount of energy. Then there is population of an energy state. It is the number of atoms per unit volume that occupy a given state of energy.

Population at each level decreases with increase of energy level.

Atoms always have thermal energies. It is distributed among available energy levels according to their energy.

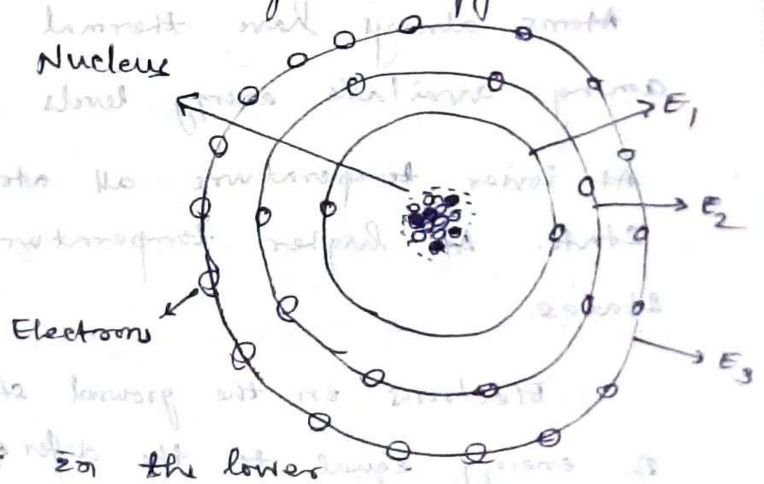
At lower temperature all atoms are in the ground state. At higher temperature atoms move to higher states.

Electrons in the ground state receives an amount of energy equal to the difference of energy of ground state and one of the excited states, absorbs energy and jumps to the excited state.

Photon of Energy $h\nu = E_2 - E_1$

Absorption, spontaneous emission and stimulated emission

* Atoms are made up of extremely small particles such as electrons, protons and neutrons.
* The strong nuclear force between protons and neutrons makes them stick together to form the nucleus.
* The electrostatic force of attraction between the nucleus and electrons causes electrons to revolve around the nucleus.
* The electrons revolving around the nucleus have different energy levels based on the distance from the nucleus.
* The electrons revolving very close to nucleus have lowest energy level.
* The electrons revolving at the farthest distance from nucleus have highest energy level.



* The electrons in the lower energy state (E_1) needs extra energy to jump into next higher energy state (E_2)
* This energy can be supplied in the form of the electric field, heat or light
* The electrons in E_1 jumps to E_2 after gaining sufficient energy from photons.

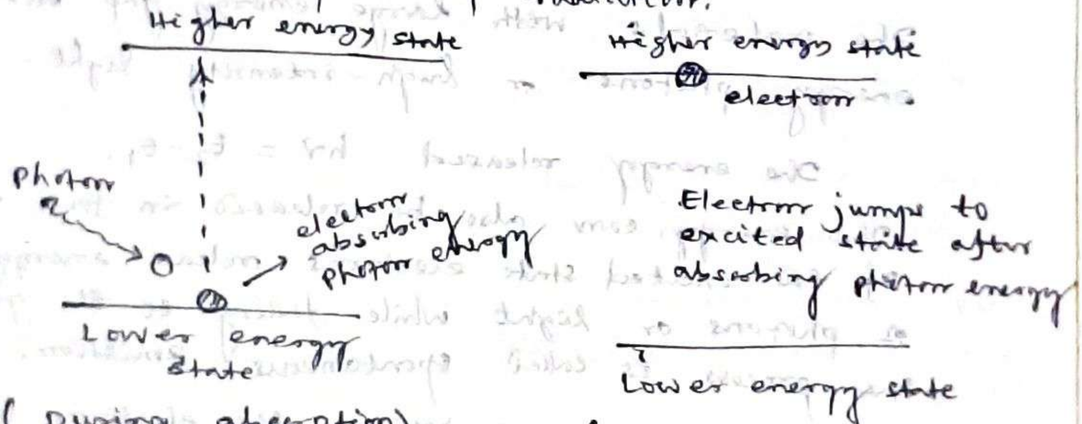
* The electrons in the higher energy state don't stay for long period. They fall back to the lower energy state after a short period losing energy in the form of light.
* The electrons in the higher energy state are known as excited electrons whereas the electrons in the lower energy state are known as ground electrons.

The light or photons interact with atoms in three ways

1) Absorption of radiation of light
2) Spontaneous emission
3) Stimulated emission.

Absorption

The process of absorbing energy from photons is called absorption of radiation.



(During absorption) (After absorption)

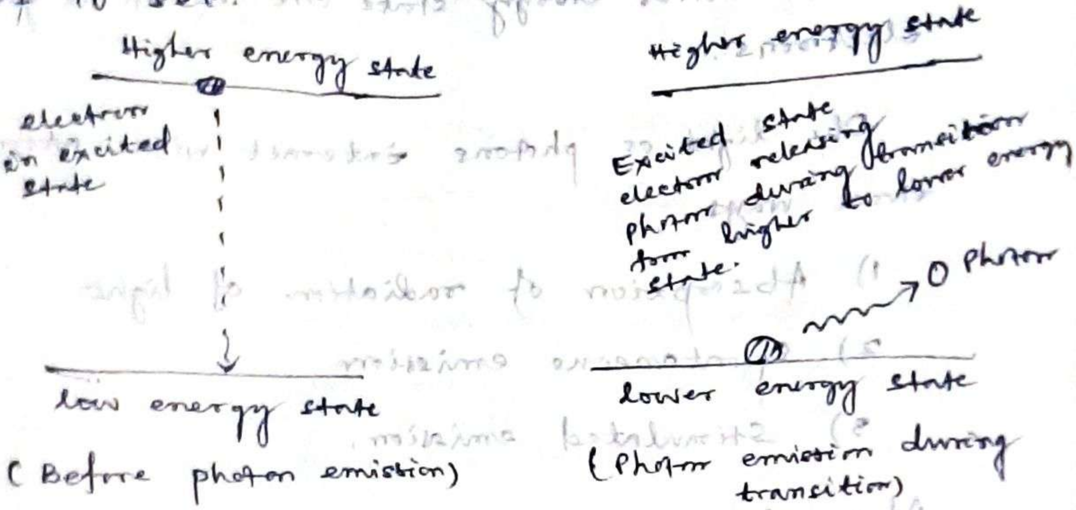
Energy absorbed $= h\nu = E_2 - E_1$.

Absorption occurs only if the energy of photon exactly matches the difference in energy between the two electron shells or orbits.

7) Spontaneous emission

The process of by which excited electrons emit photons while falling to the ground level or lower energy level is called spontaneous emission.

The lifetime of electrons in the higher energy state or excited state is very small, of the order of 10^{-8} sec.



The energy of the emitted photon is directly proportional to the energy gap of the material.
The materials with large energy gap will emit high-energy photons or high-intensity light.

The energy released $h\nu = E_2 - E_1$.
The energy can also be released in form of heat.
If the excited state electrons release energy in form of photons or light while falling to the ground state, the process is called spontaneous emission.

In spontaneous emission, the electrons changing from higher energy state to lower energy state occurs naturally. So the photon emission also occurs naturally or spontaneously.

The photons emitted due to spontaneous emission do not flow exactly in the same direction of incident photons. They flow in random direction.

If $N_2(0)$ is the number of atoms in the excited state at $t=0$, & $N_2(t)$ " " " " " at time $= t$, $\therefore N_2(t) = N_2(0)\, e^{-t/\tau_{21}}$

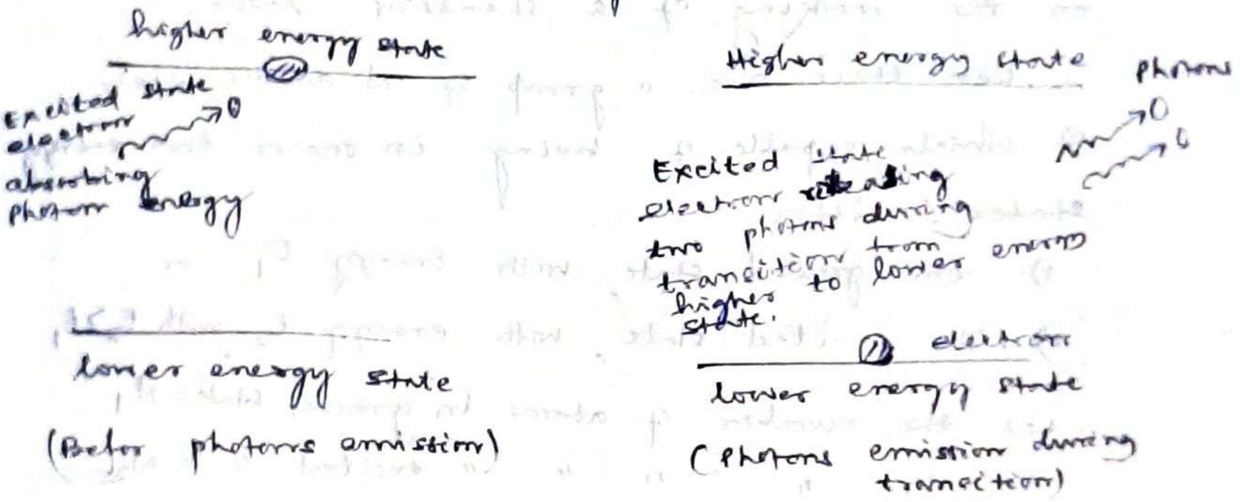
τ_{21} is the mean life time of the transition between the two states.

Stimulated emission

The process by which electrons in the excited state are stimulated to emit photons while falling to the ground state is called stimulated emission.

In this process the light/photon energy is supplied to the excited electrons in stead of supplying energy to the ground state electrons.

It is an artificial process, the electrons in the excited state need not wait for natural spontaneous emission to occur. It is an alternative method, used to stimulate excited electrons to emit photons and fall back to the ground state.



* The incident photon stimulates/forces the excited electron to emit a photon and fall into the ground state
* The energy of stimulating or incident photon must be equal to the energy difference between the two electrons shells.
* In this process, the excited electron releases an additional photon of same energy (same frequency, same phase & in the same direction).
* two photons of same energy are released.
* In this process, each incident photon generates two photons
* The photons emitted will travel in the same direction of the incident photon.
* Stimulated emission is the only method known to produce coherent light
* All photons in this process have the same frequency and travel in the same direction.

Population Inversion

A population inversion occurs while a system (group of atoms or molecules) exists in a state in which more members of the system are in higher excited states than in lower, unexcited energy states.

It is called inversion because in many familiar and commonly encountered physical systems, this is not possible. The concept is of fundamental important in laser science because the production of a population inversion is a necessary step in the working of a standard laser.

Let there are a group of N atoms, each of which capable of being in one or two energy states: either

i) the ground state, with Energy E_1 or

ii) the excited state, with energy E_2 with $E_2 > E_1$

Let the number of atoms in ground state $= N_1$

" " " " excited " $= N_2$

$$\therefore \quad N = N_1 + N_2 \longrightarrow (1)$$

The difference in energy between the two states,

$$\Delta E_{12} = E_2 - E_1 \longrightarrow (2)$$

or

$$h\nu_{12} = E_2 - E_1 \longrightarrow (3)$$

h = plank's constant

ν_{12} = frequency of light

If the group of atoms is in thermal equillibrium then their ratio is given by Boltzmann distribution

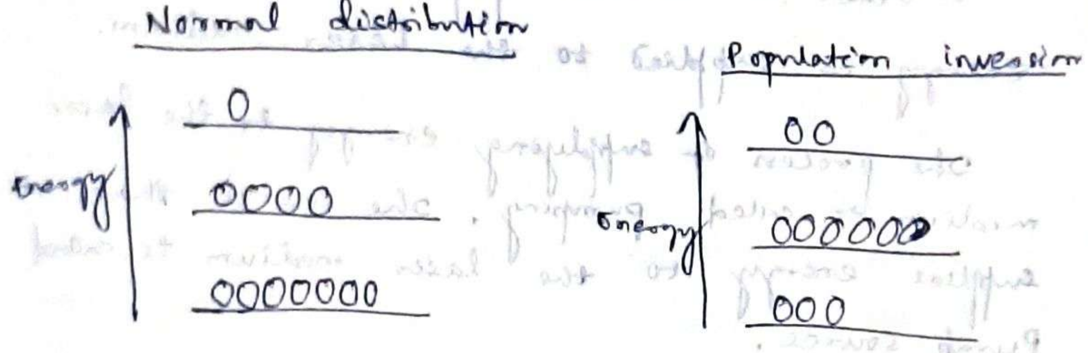
$$\frac{N_2}{N_1} = e^{\frac{-(E_2 - E_1)}{KT}} \longrightarrow (4)$$

T = thermodynamic temp.

K = Boltzmann's constant.

Under normal circumstances, the higher an energy level less it is populated by thermal energy. Under some circumstances (for example, the presence of an upper energy level that has a relatively long lifetime) a system can be constructed so that there are more atoms/molecules in an upper energy level than is allowed under conditions of normal thermodynamic equilibrium.

Such an arrangement is called population inversion.



When a population inversion exists, an upper-state system will eventually give off a photon of proper wavelength and drop to the ground state. This photon, however, can stimulate the production of other photons exactly the same wavelength because of stimulated emission of radiation.

Thus many photons of same wavelength (and phase, & other similar characteristics) can be generated in a short time.

This is light amplification by stimulated emission of radiation - LASER.

Lasers typically have a very narrow wavelength range of emission.

PUMPING MECHANISM

Under normal conditions, more electrons are there in a lower energy state than in a higher energy state.

Population inversion is the process of achieving more electrons in the higher energy state than the lower energy state.

- In other to achieve population inversion, Energy is supplied to the laser medium.

The process of supplying energy to the laser medium is called pumping. The source that supplies energy to the laser medium is called Pump source.

The type of pump sources are used depend upon laser medium. Different pump sources are used for different laser mediums to achieve population inversion.

LASER → Light Amplification by Stimulated Emission of Radiation.

Laser pumping is the act of energy transfer from an external source into the gain ~~source~~ medium of laser. The energy is absorbed in the medium, producing excited states in its atoms. When the number of particles in one excited state exceeds the number of particles in the ground state or a less excited state, population inversion is achieved.

In this condition, the mechanism of stimulated emission can take place and the medium can act as a laser or an optical amplifier.

The pump power must be higher than the lasing threshold of the laser.

The pump energy is usually provided in the form of light or electric current, but more exotic sources have been used, such as chemical or nuclear reactions.

Most commonly pump sources are

i) optical pumping

ii) Electric discharge or excitation by electrons

iii) Inelastic atom-atom collision

iv) Thermal pumping

v) chemical reactions.

Population inversion is easily achieved when the system of molecules or atoms have the energy levels with favourable properties.

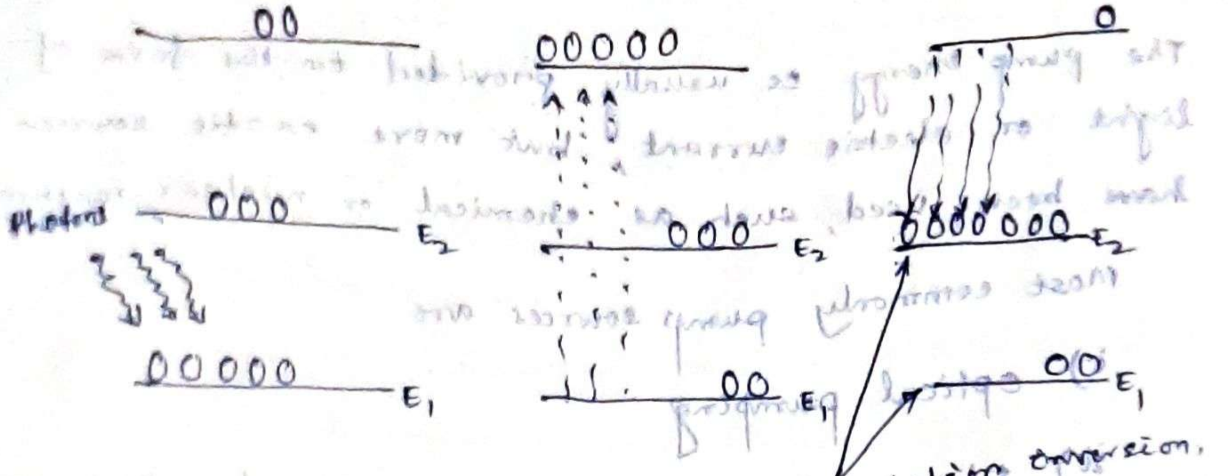
For example - the upper energy level has a long lifetime and the lower energy level has a short lifetime.

OPTICAL PUMPING

Light is used to supply energy to the laser medium in this method. An external light source like xenon flash lamp is used to produce more electrons (a high population) in the higher energy level of the laser medium.

When light source provides enough energy to the lower energy state electrons in the laser medium, they jump into higher energy state E_3. The electrons in the higher energy state do not stay for long period.

After a very short period, they fall back to the next lower energy state or meta stable state E_2 by releasing radiation less energy.



The metastable state E_2 has a greater life-time than the lower energy state or ground state E_1.

Hence more electrons are accumulated in the energy state E_2 than the lower energy state E_1.

Thus population inversion is achieved. Optical pumping is used in solid-state lasers such as Ruby lasers.

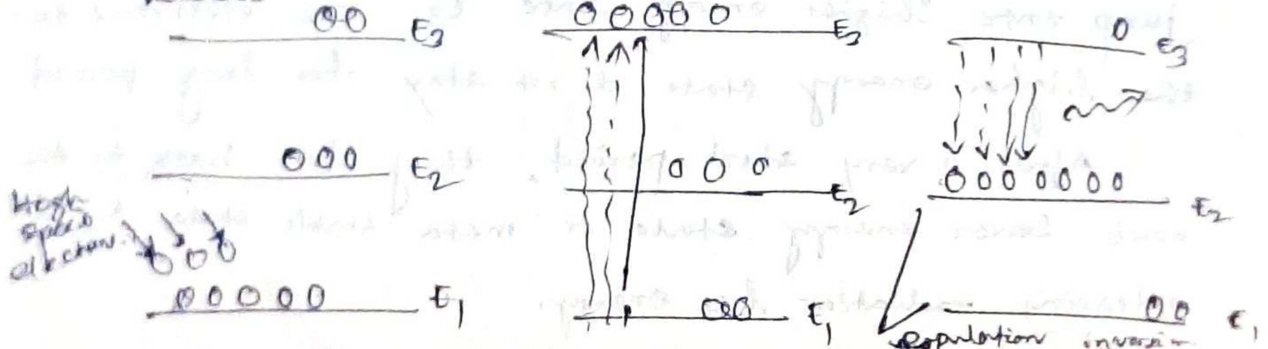
ELECTRICAL PUMPING

In this method of pumping, electric discharge (flow of electrons or electric current or electric charge through a gas, liquid or solid) acts as the pump source or energy source.

A high voltage electric discharge is passed through ~~the laser medium or gas, and electric current liquid or solid.~~ the laser medium or gas.

The intense electric field accelerates the electrons to high speeds and they collide with neutral atoms in the gas.

As a result the electrons in the lower energy state gains sufficient energy from external electrons and jumps into to higher energy state. This method of pumping is used in gas lasers such as argon lasers.



The process of achieving population inversion in the gas laser is almost similar to solid laser.

The only difference is the pump source used for supplying energy and the type of material or medium (solid or gas) used as a laser medium.

In solid lasers, an external light source like xenon flash lamp is used as pump source, whereas, in gas lasers, a high voltage electric ~~source~~ discharge is used as a pump source.

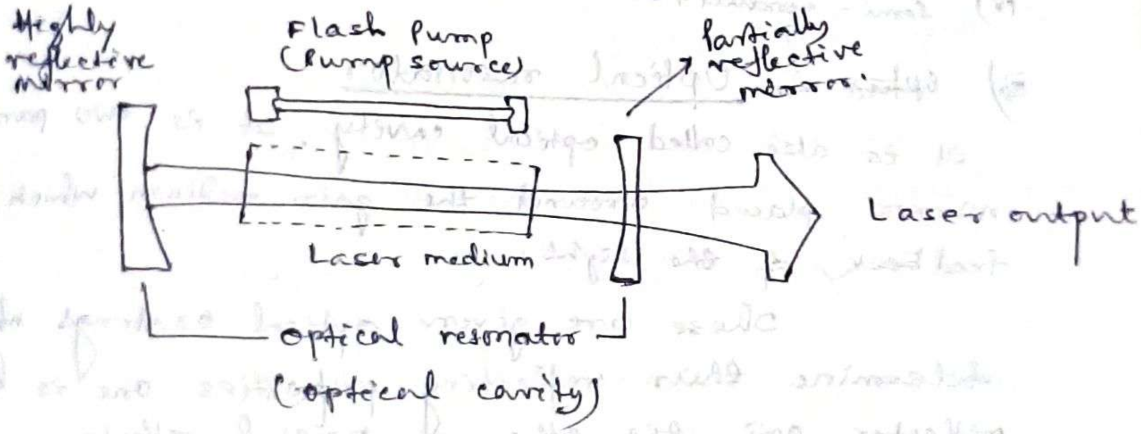
Components of a laser system

It has three principal parts

i) An energy source (Pump source)

ii) A gain medium (Laser medium)

iii) Two or more mirrors that form an optical resonator.



1) Pump source – It provides energy to the laser system

Examples: electrical discharges, flash lamps, arc lamps, light from another laser, chemical reactions, explosive devices.

It depends upon gain medium and determines how the energy is transmitted to the medium.

ii) Laser Medium

It is the major determining factor of the wavelength of operation and other properties.

Laser or gain media in different materials have linear spectra or wide spectra.

Wide spectra allows tuning of laser frequency. The gain medium is excited by the pump source to produce a population inversion and it is in the gain medium where spontaneous and stimulated emission of photons take place, leading to the phenomenon of optical gain or amplification.

Example

i) Liquids, such as ~~day~~ dye lasers: Methanol, ethanol, ethylene glycol added with chemical dyes coumarin, rhodamine, fluorescein.

ii) Gases, carbon ~~dioxe~~ dioxide, argon, krypton.

iii) Solids, such as crystals and glasses.

iv) Semi-conductors

iii) ~~Optical~~ Optical resonator

It is also called optical cavity. It is two parallel mirrors placed around the gain medium which provide feedback of the light.

These are given optical coatings which determine their reflecting properties. One is high reflector and the other is partial reflector.

The latter is called the output coupler as it allows some of the light to leave the cavity to produce the laser's output beam.

Some times 04 or more mirrors forming the cavity are used. Other optical devices such as spinning mirrors, modulators, filters and absorbers may be placed within the optical resonator to produce a variety of effects on the laser output.

Active medium (Laser active medium)

It is a collection of atoms or molecules, which can be excited into a population inversion situation. and can have electromagnetic radiation extracted out of it by stimulated emission.

It can be any state of matter
i.e. solid, liquid, gas or plasma

It determines the possible wavelengths that can be emitted from the laser.

The lists of materials that lase under certain laboratory conditions include hundreds of substances and the number increases with time.

The energy pumped into the active medium is usually highly entropic i.e very disorganised, while the resulting laser radiation is highly ordered and thus has lower entropy.

LASER → It produces lights. These lights have no existence in nature.

These lights can be produced through a process of optical amplification based on the stimulated emission of electromagnetic radiation.

* Light from Laser (contains only one colour and it is monochromatic.
* All wavelengths are in phase - It is coherent
* The light beams are very narrow & can be concentrated on one tiny spot, this property is known as collimated.

characteristics

* For operation, Population inversion is much needed.
* when a group of atoms/molecules exists with more number of electrons in an excited state than in lower energy states, Population inversion

takes place

* spontaneous emission – when an electron in excited state decay to an empty lower energy state without external influence & emitting a photon then it is called spontaneous emission.

* stimulated emission – If the electron is stimulated by a light wave (photon) to emit the second wave and return to the lower level, it is known as stimulated emission.

Practically a photon hits an electron and two photons are produced.

Applications

1) used in CD, VCD, DVD players, printers & scanners.
2) used in medicine for surgical purposes & various skin treatments
3) cutting & welding materials in industry
4) used in military & law enforcement agencies for making targets & measuring length
5) scientific research

types

1. Solid state Laser
2. Gas Laser
3. Dye or liquid laser
4. Excimer laser
5. chemical laser &
6. semiconductor laser

RUBY LASER

It is the 1st type of laser actually constructed. The ruby mineral (corundum) is aluminum oxide with a small amount of chromium which gives it its characteristics pink or red colour by absorbing green and blue light.

It is a solid state laser that uses a synthetic ruby crystal as its gain medium.

Ruby laser produce pulses of coherent visible light at a wavelength of 694.3 nm, which is a deep red colour. Typical ruby laser pulse lengths are on the order of a millisecond.

Basic concept

The design of ruby laser is made in such a way that it elliminates the reflections from the ends & external dielectric mirrors are used to form the optical cavity. Curved mirrors are typically used to relax the alignment tolerances and to form a stable resonator, often compensating for thermal lensing.

It consists of a ruby rod that must be pumped with very high energy, usually from a flash tube, to a-chieve a population inversion.

Applications

a) Military range finding - The laser with rotating prism q-switches used for military range finding.

b) in reactr research

c) used to excite laser dying

d) drilling holes in diamond.

e) used to produce holographic portraits.

f) create holograms such as aircraft tyres to look for weakness in the lining

g) are used in tatoo and hair removal.

Helium Neon laser (He-Ne) Laser

The main drawback of ruby laser is that the output beam is not contineous though very intense. He-Ne laser is used for contineous laser beam. Here the vapours of metals are employed as active media.

The output power of He-Ne laser is moderate but inferior to that of crystal lasers.

The He-Ne laser requires a mixture of two gases such that some excited level of one gas falls close enough to an excited level of 2nd gas. The gas discharge in the mixture excited the 1st gas. Now collisions transfer the excitation to 2nd gas. This produces the laser beam.

Advantage

1) These are exceptionally high monochromaticity, most pure spectrum and high stability of frequency.

Application

They have wide applications in various branches of science and engineering particularly in communication.

SEMI-CONDUCTOR LASER

In semi-conductors such as Gallium Arsenide (GaAs) the energy at the p-n junction is release because of recombination of holes & electrons near the junction region. The amount of energy released is called the activation energy or energy gap.

The energy thus released as light because the atoms of the crystals are not involved in the release of energy.

The wavelength depends upon the activation energy of the crystal.

Photons emitted at the moment of recombination of an electron with a hole will stimulate recombination of other carriers of electric charge.

The result will be stimulated emission of radiation.

* Semiconductor lasers may reach 100% efficiency.
* They are capable of ensuring a high stability of the output frequency.

Engg. application

i) Due to narrow band width lasers are used in microwave communication.

ii) Due to narrow angular speed, the laser beam has become a means of communication between earth and moon & other satellites.

iii) The distance between earth & moon can be measured by the use of lasers.

iv) By the use of lasers, the storage capacity for information in computers is greatly improved.

STRUCTURE OF OPTICAL FIBRE

Fibre optics - It is a technology related to transportation of optical energy (light energy) in guiding media specifically glass fibres.

It provides necessary wave guide for light in a media.

Principle → It is based upon principle of Total internal reflection (TIR). It is a hair thin cylindrical fibre of glass or any transparent dielectric medium. There is a very little absorption of light as it travels a very long distance through optical fibre.

It is used for optical signal transmission & for transmission and receiving electrical signals.

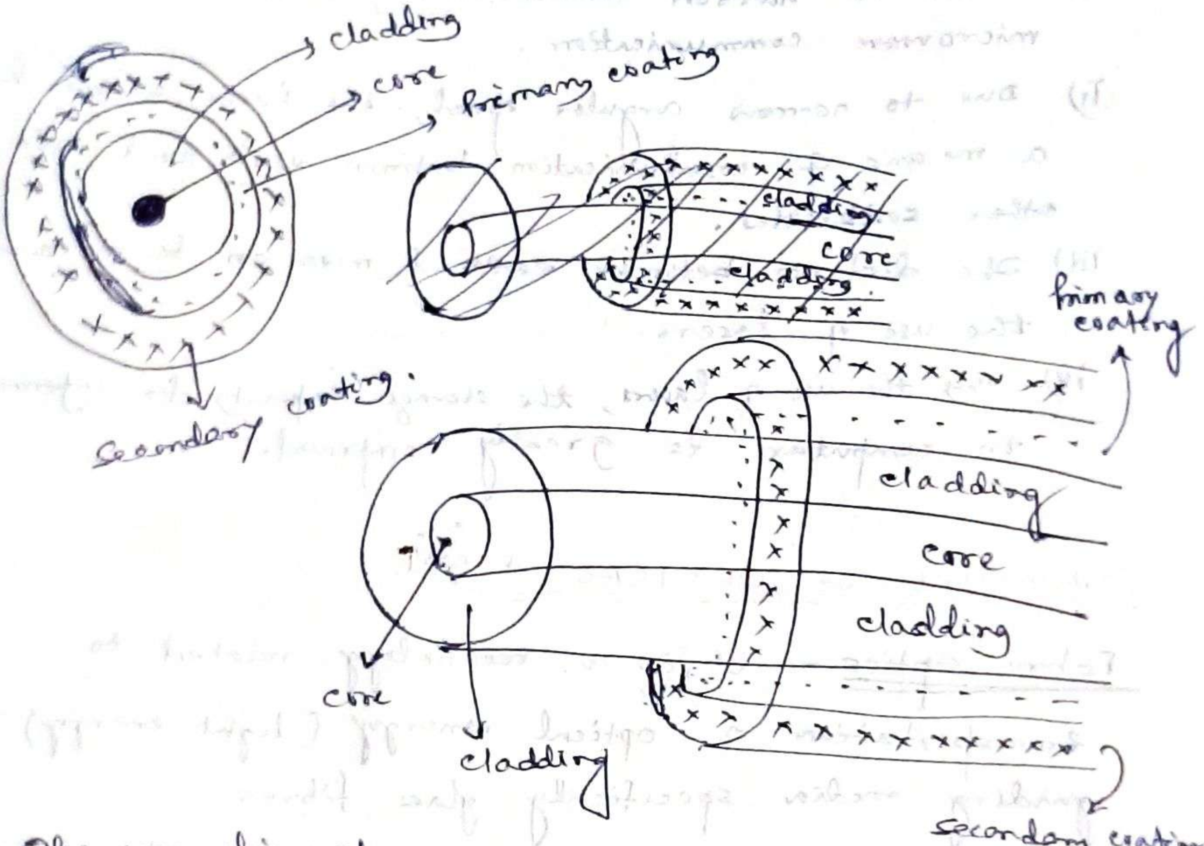
In this process the light ray is guided through the fibre from one end to other end without any energy being lost due to refraction.

Construction

It consists of an inner cylinder made of glass called core. The core carries light.

It is surrounded by another cylindrical shell of lower refractive index, called the cladding.

The cladding helps to keep the light within the core through TIR.



The core diameter → 5 μm ~ 100 μm

cladding " → 125 μm

For greater strength & protection of the fibre a soft plastic coating (Primary coating) is done → diameter → 250 μm

Secondary coating on the top.

Classification There are two types of optical fibres

i) Step-index optical fibre

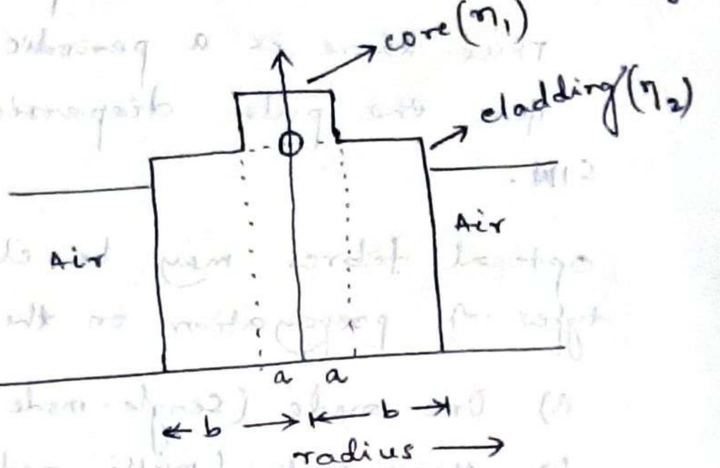
ii) Graded index optical fibre

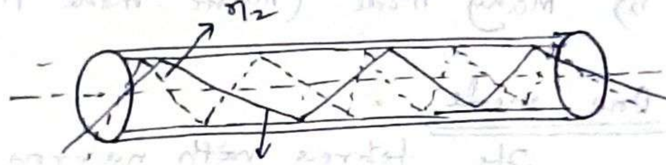
i) Step-Index (SIN) optical fibre

In this type of optical fibre the core has a uniform refractive index η_1 and the cladding has also a uniform refractive index η_2 &

$$\eta_1 > \eta_2$$

Let a & b be radius of core and cladding respectively.





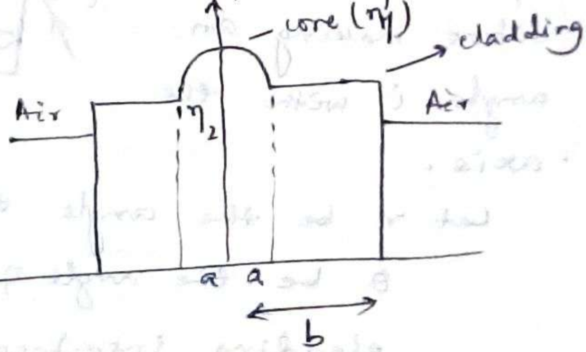
Path of two rays are drawn. The two rays travel different path lengths and emerge out at different times.

It is obvious that an input pulse gets widened as it travels along the fibre.

ii) Graded - Index Optical fibre (GRIN)

If the core has a non-uniform refractive index that gradually decreases from the centre towards the core-cladding interface, then it is called a Graded Index optical fibre.

The cladding has a uniform refractive index.



SIMPLE HARMONIC OSCILLATION

A particle or body that executes simple harmonic motion is called a simple harmonic oscillator.

Let us consider a spring oscillating along x-direction

Let one end of the spring is fixed at O and the other end is extended upto P at a distance x from O.

The restoring force acting on the other end is proportional to x

$\therefore \quad F \propto x$

$\Rightarrow \quad F = -Kx \rightarrow (1)$ where K = constant of proportionality known as force constant

K = force per unit displacement

'$-$'ve sign shows that the restoring force is opposite to the displacement.

The restoring force produces an acceleration $a = d^2x/dt^2$ on the particle.

$\therefore \quad F = \text{mass} \times \text{accl}^n = m \times d^2x/dt^2 \rightarrow (2)$

From eqn (1) & (2)

$$m\frac{d^2x}{dt^2} = -Kx$$

$$\Rightarrow \frac{d^2x}{dt^2} + \frac{K}{m}x = 0$$

$$\Rightarrow \boxed{\frac{d^2x}{dt^2} + w^2x = 0} \longrightarrow (3)$$

Where $\omega^2 = \frac{k}{m}$ → (4)

This is the differential eq^n of SHM.

The general sol^n to this eq^n is

$$x = c e^{at} \longrightarrow (5)$$

where c & a are constants.

Differentiating eq^n (5)

$$\frac{dx}{dt} = ace^{at} \text{ and } \frac{d^2x}{dt^2} = a^2ce^{at} \longrightarrow (6)$$

Substituting in eq^n (3)

$$a^2ce^{at} + \omega^2 ce^{at} = 0$$

$$\Rightarrow (a^2+\omega^2) ce^{at} = 0$$

$$\Rightarrow a^2 + \omega^2 = 0 \quad \text{since } c \neq 0 \ \& \ e^{at} \neq 0$$

$$\Rightarrow a^2 = -\omega^2$$

$$\Rightarrow a = \sqrt{-\omega^2} = \pm i\omega \longrightarrow (8)$$

using the value of a in general solutions

$$x = c_1 e^{i\omega t} + c_2 e^{-i\omega t} \longrightarrow (9)$$

where c_1 & c_2 are constants.

Expanding

$$x = c_1(\cos\omega t + i\sin\omega t) + c_2(\cos\omega t - i\sin\omega t)$$

$$\Rightarrow x = (c_1+c_2)\cos\omega t + i(c_1-c_2)\sin\omega t$$

$$\Rightarrow x = a\sin\phi\cos\omega t + a\cos\phi\sin\omega t \longrightarrow (11)$$

where $c_1 + c_2 = a\sin\phi$

$i(c_1 - c_2) = a\cos\phi$

where a & ϕ are constants

or $\boxed{x = a \sin(\omega t + \phi)}$ ⟶ (11)

This is the solution for a SHM.

Here x = displacement of the particle

a = amplitude of oscillation.

(i) Velocity of SHM

$$V = \frac{dx}{dt} = \omega a \cos(\omega t + \phi) \longrightarrow (12)$$

From eqn (11) $\sin(\omega t + \phi) = \frac{x}{a}$

$$\cos(\omega t + \phi) = \sqrt{1 - \sin^2(\omega t + \phi)} = \sqrt{1 - \frac{x^2}{a^2}}$$

$$\Rightarrow \cos(\omega t + \phi) = \frac{1}{a}\sqrt{a^2 - x^2} \longrightarrow (13)$$

Putting eqn (13) in (12)

$$\boxed{V = \omega\sqrt{a^2 - x^2}} \longrightarrow (14)$$

At mean position $(x = 0)$, the velocity is maximum.

At the extreme position $(x = \pm a)$, the velocity of the particle is zero.

(ii) Time Period (T)

The time taken by the particle to complete one oscillation.

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{x}{d^2x/dt^2}} \quad \text{from eqn (9)}$$

$$\boxed{T = 2\pi\sqrt{\frac{\text{displacement}}{\text{acceleration}}}} \longrightarrow (15)$$

(iii) Frequency (n)

The number of oscillations performed in one second is called frequency.

$$n = \frac{1}{T} = \frac{\omega}{2\pi} = \boxed{\frac{1}{2\pi}\sqrt{\frac{K}{m}}} \longrightarrow (16)$$

Alternatively

From eqⁿ (2)

$$\frac{d^2 y}{dt^2} + \frac{K}{m} y = 0 \rightarrow ① \quad (y = \text{displacement})$$

$$\Rightarrow \frac{d^2 y}{dt^2} + w^2 y = 0 \rightarrow ② \quad \text{where } w = \sqrt{\frac{K}{m}}$$

which depends upon mass & nature of the medium.

Multiplying both sides by $2\frac{dy}{dt}$

$$\Rightarrow 2\frac{dy}{dt}\frac{d^2y}{dt^2} = -w^2 2y \frac{dy}{dt}$$

$$\Rightarrow \frac{d}{dt}\left[\frac{dy}{dt}\right]^2 = -\frac{d}{dt}(w^2 y^2) \rightarrow ③.$$

Integrating with respect to t

$$\left(\frac{dy}{dt}\right)^2 = -w^2 y^2 + c \rightarrow ④ \quad (c = \text{constant of Integration})$$

When $y = \pm A$ (amplitude) max.

$\frac{dy}{dt} = 0$ From (eqⁿ ④)

$$0 = -w^2 A^2 + c$$

$$\Rightarrow c = w^2 A^2 \rightarrow ⑤$$

from eqⁿ 4

$$\left(\frac{dy}{dt}\right)^2 = -w^2 y^2 + w^2 A^2 = w^2(A^2 - y^2)$$

$$\Rightarrow \frac{dy}{dt} = w\sqrt{A^2 - y^2} \rightarrow ⑥$$

$$\Rightarrow \frac{dy}{\sqrt{A^2-y^2}} = \omega\, dt$$

Integrating

$$\sin^{-1}\left(\frac{y}{A}\right) = \omega t + \phi$$ (where ϕ = constant of integration)

$$\Rightarrow y/A = \sin(\omega t + \phi)$$

$$\Rightarrow \boxed{y = A\sin(\omega t+\phi)} \longrightarrow (7)$$

This is the instantaneous displacement of the body vibrating in SHM.

Here

A = Amplitude

ϕ = initial phase of vibration.

It gives the information about the particle when we started measuring time

(i) If we started measuring time from the instant the particle leaves mean position, in +ve direction,

$\phi = 0$, hence $y = A\sin\omega t \rightarrow (8)$

(ii) If the particle is at +ve extremity

$\phi = \pi/2$, $y = A\sin(\omega t + \pi/2)$

$$\Rightarrow y = A\cos\omega t \longrightarrow (9)$$

(iii) If the particle leaves the mean position in −ve direction

$\phi = \pi$, $y = A\sin(\omega t + \pi)$

$$\Rightarrow y = -A\sin\omega t \longrightarrow (10)$$

(iv) If the particle is at −ve extemity,

$\phi = 3\pi/2$, $y = A\sin(\omega t + 3\pi/2)$

$$\Rightarrow y = -A\cos\omega t \longrightarrow (11)$$

Characteristics of SHM

(i) **Displacement (y)**

It is the distance of the particle from mean position at a particular instant

$$y = A\sin(\omega t + \phi) \longrightarrow (13)$$

(ii) **Amplitude (A)**

It is the maximum displacement of a particle on either side of the mean position.

$$A = y / \sin(\omega t + \phi) \longrightarrow (14)$$

(iii) **Time period (T)**

The smallest time interval after which an oscillating particle passes through same condition.

$$T = \frac{2\pi}{\omega} \rightarrow (15) \quad (\omega = \text{angular frequency})$$

If we put $t + \frac{2\pi}{\omega}$ in place of t in eqn (7)

$$A\sin\left(\omega\left(t + \frac{2\pi}{\omega}\right) + \phi\right) = A\sin(2\pi + \omega t + \phi)$$

$$= A\sin(\omega t + \phi) = y$$

The value of y repeats after a time $\frac{2\pi}{\omega}$

$$\therefore \boxed{T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\frac{K}{m}}} = 2\pi\sqrt{\frac{m}{K}}} \longrightarrow (16)$$

(iv) **Frequency (n)** → It is the number of vibrations completed in one second.

$$n = \frac{1}{T} = \frac{1}{2\pi\sqrt{\frac{m}{K}}} = \frac{1}{2\pi}\sqrt{\frac{K}{m}}$$

$$\boxed{n = \frac{1}{2\pi}\sqrt{\frac{K}{m}}} \longrightarrow (17)$$

(v) Phase (ϕ)

To have complete knowledge about a particle undergoing SHM, the following information must be known.

i) Position of the particle

ii) Direction of motion.

Phase gives us both the above information.

Hence, Phase of a particle is defined as its state as regards its position and direction of motion.

ϕ is also known as phase constant

at $t = 0$, $y = A \sin\phi$. (constant)

(vi) Velocity (V)

$$V = \frac{dy}{dt} = \frac{d}{dt}[A\sin(wt+\phi)] = Aw\cos(wt+\phi) \rightarrow (18)$$

$$\cos(wt+\phi) = \sqrt{1-\sin^2(wt+\phi)} = \sqrt{1-(y/A)^2}$$

$$= \sqrt{\frac{A^2-y^2}{A^2}} = \frac{1}{A}\sqrt{A^2-y^2} \rightarrow (19)$$

$$\therefore V = Aw\left(\frac{1}{A}\sqrt{A^2-y^2}\right) = w\sqrt{A^2-y^2}$$

$$\therefore \boxed{V = w\sqrt{A^2-y^2}} \longrightarrow (20)$$

At $y = 0$, $V = \pm Aw$ (Maximum) mean position

At $y = \pm A$ (extreme position)

$V = 0$ (no motion) (extreme position)

A particle undergoing SHM possesses maximum velocity at mean position and is at rest while at the extreme position.

(vii) Acceleration (a)

From eqⁿ (18)

$$V = A\omega \cos(\omega t + \phi)$$

$$a = \frac{dv}{dt} = -\omega^2 A \sin(\omega t + \phi) = -\omega^2 y \rightarrow (20)$$

$$\Rightarrow \boxed{a = -\omega^2 y} \longrightarrow (21)$$

At $y = 0$, $a = 0$

at $y = \pm A$, $a = \mp \omega^2 A$

A particle has zero acceleration while passing through the mean position and has maximum acceleration while at extreme position.

(viii) Kinetic Energy (E_K)

It is the energy possessed by a body by virtue of its motion.

$$E_K = \frac{1}{2} m v^2 = \frac{1}{2} m \left(\omega\sqrt{A^2 - y^2}\right)^2$$

$$\therefore E_K = \frac{1}{2} m \omega^2 \left(\sqrt{A^2 - y^2}\right)^2$$

$$\therefore E_K = \frac{1}{2} m \omega^2 (A^2 - y^2) \longrightarrow (22)$$

At $y = 0$, $E_K = \frac{1}{2} m \omega^2 A^2$ (maximum)

At $y = \pm A$, $E_K = 0$ (minimum)

Kinetic Energy is maximum at the mean position and minimum or zero at the extreme position.

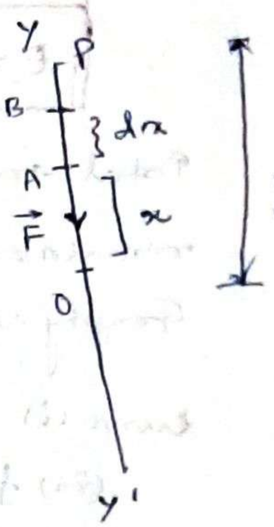
(ix) Potential Energy (PE)

It is the energy possessed by the particle by virtue of its position.

In SHM, the work done in removing the particle to a position away from the mean position is called potential energy.

(21)

Let A be the position of the body at any instant at a distance x from O. Let m be the mass of the body



Acceleration at A $= -\omega^2 x$ (eq^n - 21)

Force at A is

$$\vec{F} = m(-\omega^2 x)$$

If dw is the amount of work done in displacing it through a distance $\vec{dx}$,

$$dw = \vec{F} \cdot \vec{dx} = F dx \cos 180^\circ = (-m\omega^2 x) dx(-1)$$

$$dw = m\omega^2 x \, dx$$

Total workdone in removing the particle from O to P is

$$W = \int_0^w dw = \int_0^y m\omega^2 x \, dx = \left. \frac{m\omega^2 x^2}{2} \right]_0^y$$

$$\therefore W = \frac{1}{2} m\omega^2 y^2 \longrightarrow (23)$$

This is actually the potential Energy possessed by the body.

Hence

$$E_p = \frac{1}{2} m\omega^2 y^2 \longrightarrow (24)$$

(*) Total Energy (E)

Total energy E of the particle, at any instant of time, is the sum total of instantaneous E_K & E_P.

$$E = E_K + E_P = \frac{1}{2} m w^2 (A^2 - y^2) + \frac{1}{2} m w^2 y^2$$

$$\boxed{E = \frac{1}{2} m w^2 A^2} \longrightarrow (25)$$

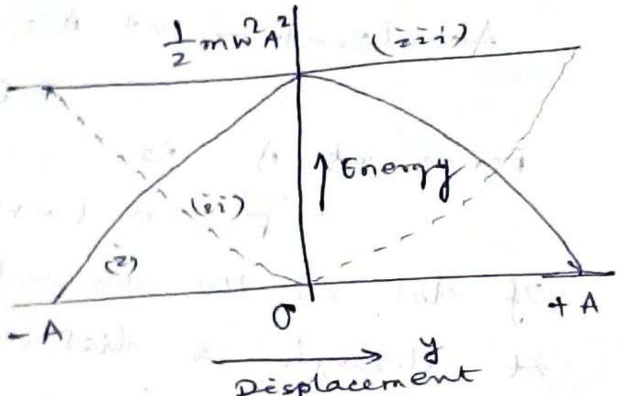
Total energy of a particle executing SHM remains constant. as m, w, A are constants.

Graphically

curve (i) for E_K

(ii) for E_P

(iii) for E



When $y = \pm \frac{1}{\sqrt{2}} A$

$$E_K = \frac{1}{2} m w^2 \left(A^2 - \frac{1}{2} A^2 \right) = \frac{1}{4} m w^2 A^2$$

$$E_P = \frac{1}{2} m w^2 y^2 = \frac{1}{2} m w^2 \frac{1}{2} A^2 = \frac{1}{4} m w^2 A^2$$

$$E_K = E_P.$$

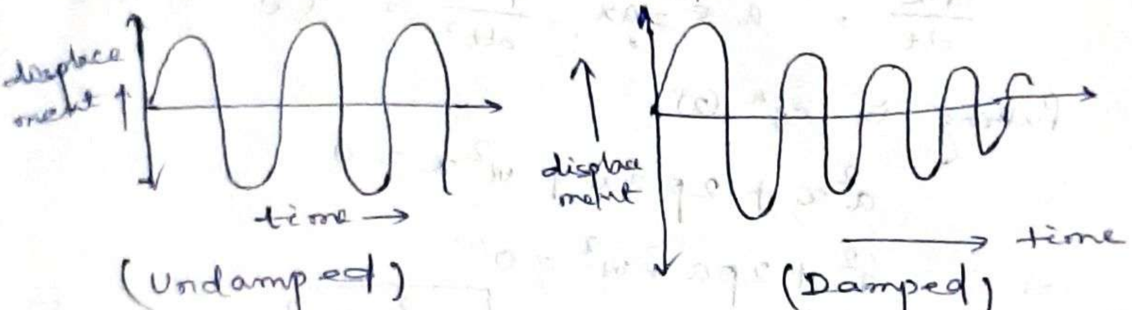
Here E_K & E_P are known as average energies.

Average value of E_K or $E_P = \frac{1}{2}$ of total energy.

DAMPED HARMONIC OSCILLATION

The vibrations in which the amplitude decreases gradually are called damped vibrations. Such vibrations of decaying amplitude occurs because of number of frictional forces (external or internal) act on the particle.

Example – Motion of simple pendulum.



In case of damped vibration, two types of forces act on the vibrating body

a) Restoring force $\propto$ dispacement which tends to bring the particle back to its original position, $F_r = -Kx$.

b) Damping or retarding force $\propto$ velocity

$$F_d = -bv$$

where b = ~~damping~~ coefficient ~~or~~ resistive force caused by friction per unit velocity

$\therefore$ Net force $F = F_r + F_d = -Kx - bv$

$\Rightarrow \quad ma = -Kx - bv \qquad (F = \text{mass} \times \text{acc}^n)$

$$\Rightarrow \quad m\frac{d^2x}{dt^2} = -Kx - b\frac{dx}{dt} \longrightarrow (1)$$

$$\Rightarrow \quad \frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{K}{m}x = 0$$

$$\Rightarrow \quad \boxed{\frac{d^2x}{dt^2} + 2\beta\frac{dx}{dt} + w^2x = 0} \longrightarrow (2)$$

where $\frac{b}{m} = 2\beta$ & $\frac{K}{m} = w^2$

Eq^n (2) represents the differential equation of a damped harmonic oscillator.

& $\beta = \frac{b}{2m}$ is the damping co-efficient

Let the solution to eq^n (2) be

$$x = e^{at}$$

$$\frac{dx}{dt} = a e^{at} = ax, \quad \frac{d^2x}{dt^2} = a^2 e^{at} = a^2 x$$

Putting in eq^n (2).

$$a^2 x + 2\beta a x + w^2 x = 0$$

$$\Rightarrow a^2 + 2\beta a + w^2 = 0$$

$$\Rightarrow a = \frac{-2\beta \pm \sqrt{4\beta^2 - 4w^2}}{2} = -\beta \pm \sqrt{\beta^2 - w^2}$$

The possible solutions can be

$$x = A e^{(-\beta + \sqrt{\beta^2 - w^2})t}$$

$$x = B e^{(-\beta - \sqrt{\beta^2 - w^2})t}$$

The general solution shall be

$$x = A e^{(-\beta + \sqrt{\beta^2 - w^2})t} + B e^{(-\beta - \sqrt{\beta^2 - w^2})t}$$

where A & B are constants, determined from the boundary conditions.

$$\Rightarrow x = e^{-\beta t}\left(A e^{(\sqrt{\beta^2 - w^2})t} + B e^{(-\sqrt{\beta^2 - w^2})t}\right) \quad \text{--- (4)}$$

At $t = 0$, $x = x_0$ at extreme position

similarly at $t = 0$, $\frac{dx}{dt} = 0$ at extreme position

$$\therefore A = \frac{x_0}{2}\left[1 + \frac{\beta}{\sqrt{\beta^2 - w^2}}\right] \longrightarrow (5)$$

$$B = \frac{x_0}{2}\left[1 - \frac{\beta}{\sqrt{\beta^2 - w^2}}\right] \longrightarrow (6)$$

The following three cases of motion may occur,

Case-I Heavy damping / Over damped / dead beat motion

When $\beta > \omega$ i.e. damping constant > restoring constant

$\sqrt{\beta^2 - \omega^2} < \beta$ (a real quantity)

Let $\sqrt{\beta^2 - \omega^2} = \gamma$

& $A + B = C$, $A - B = D$

$\therefore$ $A = \frac{C+D}{2}$, $B = \frac{C-D}{2}$

Putting in eq^n (4)

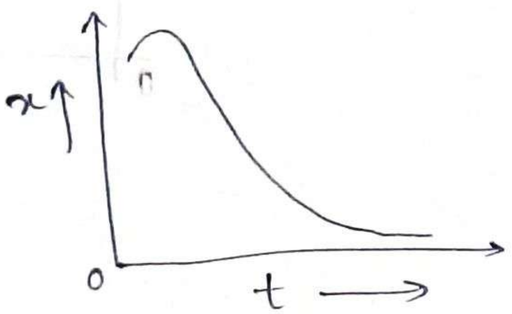
$$x = e^{-\beta t}\left(\left(\frac{C+D}{2}\right)e^{\gamma t} + \left(\frac{C-D}{2}\right)e^{-\gamma t}\right)$$

$$\Rightarrow x = e^{-\beta t}\left[C\left(\frac{e^{\gamma t} + e^{-\gamma t}}{2}\right) + D\left(\frac{e^{\gamma t} - e^{-\gamma t}}{2}\right)\right]$$

$$\Rightarrow \boxed{x = e^{-\beta t}\left[C \cosh \gamma t + D \sinh \gamma t\right]} \rightarrow (7)$$

This equation represents a motion which is non oscillatory or aperiodic, dead beat or overdamped motion.

In this case, displacement after passing through its first maximum decays asymptotically to zero



Examples
a) motion of pendulum in viscous liquid
b) dead beat moving coil galvanometer
c) Discharging of capacitor in a C-R circuit

Case-II critical damping

When $\beta \to \omega$, $\gamma = \sqrt{\beta^2 - \omega^2}$ is a very small quantity

$$x = e^{-\beta t}\left(A e^{\gamma t} + B e^{-\gamma t}\right) \quad (\text{from eq}^n (4))$$

Hence neglecting the small quantities

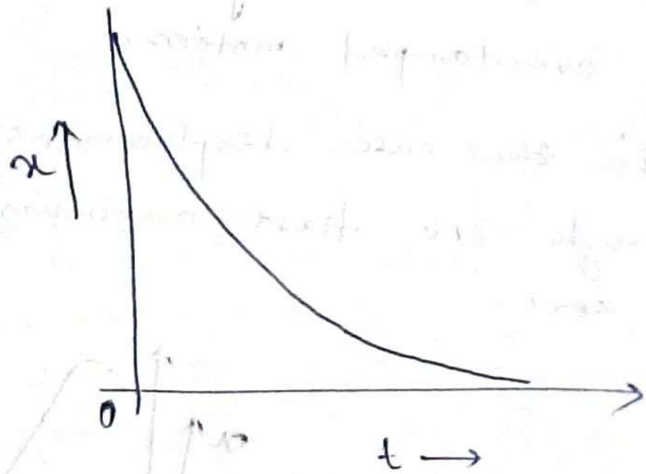
$$x = e^{-\beta t}\left(A(1+\gamma t) + B(1-\gamma t)\right)$$

$$\Rightarrow x = e^{-\beta t}\left((A+B) + (A-B)\gamma t\right)$$

$$\Rightarrow x = e^{-\beta t}\left(P + Qt\right) \rightarrow (8)$$

where $P = A+B$, $Q = (A-B)\gamma$

This is an equation which represents decrease of amplitude exponetially. This is the transition case and is known as the case of critical damping. The particle tends to move to its equilibrium position much more rapidly than the overdamped motion. The motion is just aperiodic & non-oscillatory



Case - III Light damping or underdamped motion

when $\beta^2 < \omega^2$

then $\sqrt{\beta^2 - \omega^2} = i\gamma$, $\gamma = \sqrt{\omega^2 - \beta^2}$

From eqn (4)

$$x = e^{-\beta t}\left(Ae^{i\gamma t} + Be^{-i\gamma t}\right)$$

If $A = \frac{C+D}{2}$ & $B = \frac{C-D}{2}$

$$\therefore x = e^{-\beta t}\left(\left(\frac{C+D}{2}\right)e^{i\gamma t} + \left(\frac{C-D}{2}\right)e^{-i\gamma t}\right)$$

$$= e^{-\beta t}\left[C\left(\frac{e^{i\gamma t} + e^{-i\gamma t}}{2}\right) + D\left(\frac{e^{i\gamma t} - e^{-i\gamma t}}{2}\right)\right]$$

$$\therefore x = e^{-\beta t}\left[C\cos\gamma t + Di\sin\gamma t\right]$$

taking $C = P\sin\phi$ & $Di = P\cos\phi$

$$\therefore x = e^{-\beta t}\left(P\sin\phi\cos\gamma t + P\cos\phi\sin\gamma t\right)$$

$$\Rightarrow \boxed{x = Pe^{-\beta t}\sin(\gamma t + \phi)} \longrightarrow (9)$$

Eqn (9) represents damped oscillatory motion of frequency n. Here the constants P & ϕ are determined from initial position & velocity of oscillator.

$$n = \frac{\gamma}{2\pi} = \frac{\sqrt{\omega^2 - \beta^2}}{2\pi} = \frac{\omega}{2\pi}\sqrt{1 - \frac{\beta^2}{\omega^2}}$$

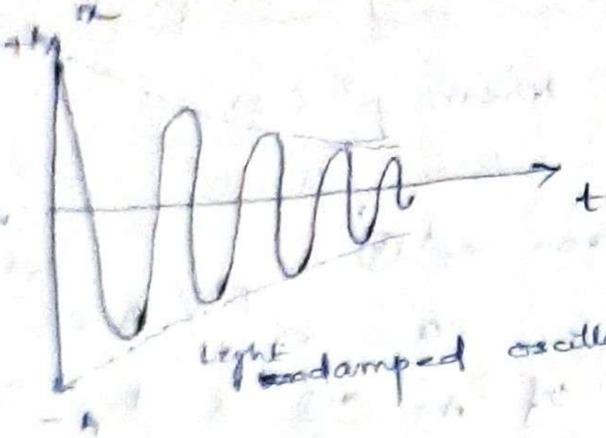
$$\Rightarrow n = n_0\sqrt{1 - \frac{\beta^2}{\omega^2}} \quad \text{where } n_0 = \frac{\omega}{2\pi}$$

n_0 = natural frequency i.e frequency without damping. The frequency of damped oscillator is less than that of undamped oscillator.

The amplitude of the motion $\propto e^{-\beta t}$

$e^{-\beta t}$ is called the damping factor.

Damping has two effects on oscillatory motion



Light damped oscillatory motion.

i) the amplitude ($P e^{-\beta t}$) decreases exponentially with time

ii) The decrease of frequency (n) of vibration of the body with increase of period of oscillation.

Example- a) Pendulum in air

b) the electric oscillation of LCR circuit

Decrement

The ratio between two successive amplitudes of maxima

$$\frac{P e^{-\beta t}}{P e^{-\beta(t+T)}} = e^{\beta T}$$ is the decrement of the oscillator.

$$\lambda = \ln e^{\beta T} = \beta T = \beta\left(\frac{2\pi}{\sqrt{w^2-\beta^2}}\right) = \frac{2\pi\beta}{\sqrt{w^2-\beta^2}}$$

is called Logarithmic decrement of the oscillator.

Total energy: $E = \frac{1}{2} a^2 k e^{-2\beta t}$

Power dissipation $= -\frac{dE}{dt} = +2\beta E$

Quality factor $= 2\pi \dfrac{\text{Energy stored in the system}}{\text{Energy lost per period}} = 2\pi \dfrac{E}{PT}$

And $P = \frac{E}{\tau}$ (τ = relaxation time)

$Q = 2\pi \frac{\tau}{T} = w \times \tau$ = angular frequency × relaxation time

Forced vibration

An external periodic force is supplied to the system so that the particle will vibrate regularly without decaying its amplitude. This vibration is called forced vibration. One has to apply external periodic force to maintain the oscillation.

Suppose a particle of mass m executing damped harmonic motion is subjected to an external periodic force $F_0 e^{ipt}$

where p = cyclic frequency of periodic force

Let x be the instantaneous displacement from the rest.

eq^n of forced vibration can be written as

$$m\frac{d^2x}{dt^2} = -kx - bv + F_0 e^{ipt}$$

$$\Rightarrow \boxed{\frac{d^2x}{dt^2} + 2\beta\frac{dx}{dt} + w^2x = f e^{ipt}} \longrightarrow (1)$$

where $\beta = \frac{b}{2m}$, $w^2 = \frac{k}{m}$ & $f = \frac{F_0}{m}$ i,e amplitude of driving force per unit mass.

β = damped constant $\quad w = \sqrt{\frac{k}{m}}$ = natural angular frequency.

This is a 2nd degree 1st order differential eq^n.

Its solution consists of two parts

(i) P.I (Particular integral)

(ii) CF (complementary function).

(i) PI

Let $x = A'e^{ipt}$, $\frac{dx}{dt} = ipA'e^{ipt}$

$$\frac{d^2x}{dt^2} = -p^2A'e^{ipt}$$

Substituting we get

$$-p^2 A' e^{ipt} + ip\,2\beta A' e^{ipt} + w^2 A' e^{ipt} = f e^{ipt}$$

$$\Rightarrow A'(-p^2 + 2i\beta p + w^2) = f$$

$$\Rightarrow A' = \frac{f}{w^2 - p^2 + i\,2\beta p} \longrightarrow (2)$$

Let $w^2 - p^2 = B\cos\phi$, $2\beta p = B\sin\phi$

$$\Rightarrow A' = \frac{f}{B\cos\phi + iB\sin\phi} = \frac{f}{Be^{i\phi}} \longrightarrow (3)$$

Here $B^2 = [(w^2-p^2)^2 + 4\beta^2 p^2]$

$$B = \sqrt{(w^2-p^2)^2 + 4\beta^2 p^2} \longrightarrow (4)$$

& $$\tan\phi = \frac{2\beta p}{w^2 - p^2} \longrightarrow (4a)$$

$$x = A' e^{ipt} = \frac{f}{\sqrt{(w^2-p^2)^2 + 4\beta^2 p^2}}\, e^{i(pt-\phi)} \longrightarrow (5)$$

Hence

$$\text{amplitude} = \frac{f}{\sqrt{(w^2-p^2)^2 + 4\beta^2 p^2}} \longrightarrow (6)$$

(ii) CF

CF is obtained by putting RHS of eq^n (1) = 0

$$\frac{d^2x}{dt^2} + 2\beta\frac{dx}{dt} + w^2 x = 0$$

when damping constant β is small, the solution of this damped oscillatory motion is

$$x = Ce^{-\beta t}\sin(\gamma t + \delta)$$ from eq^n (9)

Here $P = C$ & $\phi = \delta$

Here C & δ are constants, depend upon the initial conditions, &

$$\gamma = \sqrt{w^2 - \beta^2} = \text{angular frequency}$$

∴ The solution is

$x = CF + PI$

$$\therefore \boxed{x = C e^{-\beta t} \sin(\alpha t + \delta) + \frac{f}{\sqrt{4\beta^2 p^2 + (\omega^2 - p^2)^2}} e^{i(pt - \phi)}} \longrightarrow (7)$$

The 1st part (i.e. transient term) of the solution for x represents damped simple Harmonic motion.

After a few time, the vibration becomes negligible as its amplitude diminishes exponentially with time.

Thus after a lapse of time, the second part (steady state term) represents the forced sustained oscillation whose frequency $(p/2\pi)$ will be the frequency of the applied periodic force.

The steady state term describes the behaviour of the oscillator after the transient term has died away.

Hence solution for the forced vibration can be written as

$$\boxed{x = \frac{f}{\sqrt{4\beta^2 p^2 + (\omega^2 - p^2)^2}} e^{i(pt - \phi)}} \longrightarrow (8)$$

Here amplitude $A = \dfrac{f}{\sqrt{4\beta^2 p^2 + (\omega^2 - p^2)^2}}$

frequency $= p/2\pi$

phase $\phi = \tan^{-1}\left(\dfrac{2p\beta}{\omega^2 - p^2}\right) \longrightarrow (8a)$

If the physical excitations are $F_0 \cos pt$ or $F_0 \sin pt$, the physical solutions are extracted from the complex solution by taking the real & imaginary part respectively.

When the driving force is $F_0 \sin pt$, the steady state solution for the forced vibration is

$$x = \frac{f}{\sqrt{4\beta^2 p^2 + (\omega^2 - p^2)^2}} \sin(pt - \phi) \longrightarrow (9)$$

$$\Rightarrow \boxed{x = A \sin(pt - \phi)} \longrightarrow (10)$$

The phase difference between the displacement of driven oscillator and driving force depends upon the frequency of the applied force and damping co-efficient β of the medium.

In steady state condition the displacement x of driven oscillator lags behind the driving force ($F = F_0 \sin pt$) by an angle ϕ.

Velocity:

The velocity of the body executing forced vibration i/e driven oscillator is,

$$V = \frac{dx}{dt} = Ap\cos(pt - \phi) = Ap\sin(pt - \phi + \pi/2)$$

It indicates that the phase of the velocity of driven oscillator is $(pt - \phi + \pi/2)$.

→ So in steady state condition, the velocity of the driven oscillator at any point of time leads the displacement by $\pi/2$.