A set of mn numbers (real or complex) arranged in the form of a rectangular array having 'm' rows & 'n' columns is called an m × n matrix. [to be read as m by n matrix].

It is usually written as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

It is represented as $A = [a_{ij}]m \times n$

The numbers a11, a_{12}... etc are called elements of the matrix. The element aij belongs to ith row and jth column.

Special Types of Matrices

(i) Square Matrix

An m × n matrix for which m = n(i.e. the number of rows is equal to the number of columns) is called a square matrix of order n. The element aij of a square matrix A = [aij]n × n for which i = j i.e., the elements a11, a22,...,ann are called the diagonal elements.

The matrix

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 2 & 3 & 1 & 0 \\ 5 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$ is a square matrix of order 4.

The elements 0, 3, 1, 2 are the diagonal elements of A.

(ii) Null Matrix or Zero Matrix

The m × n matrix whose all elements are zero is called a null matrix of order m × n. It is usually denoted by O

(iii) Unit Matrix or Identity Matrix

A square matrix each of whose principal diagonal element is '1' and each of whose non-diagonal element is equal to zero is called a unit matrix or an identity matrix and is denoted by I. In will denote a unit matrix of order n

e.g. I3 = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ & I2 = $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(iv) Scalar Matrix

A diagonal matrix whose diagonal elements are all equal is called a scalar matrix

e.g. A = $\begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$

(v) Diagonal Matrix

A square matrix A = [aij]n × n is called a diagonal matrix if aij = 0 for all i ≠ j, i.e.,

Adjoint of a Square Matrix

Let $A = [a_{ij}]n \times n$ be any $n \times n$ matrix. The transpose B' of the matrix $B = [A_{ij}]n \times n$, where Aij denotes the cofactor of the element aij in the determinant |A|, is called the adjoint of the matrix A and is denoted by the symbol Adj A.

Thus the adjoint of a matrix A is the transpose of the matrix formed by the

cofactors of A i.e. if

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \text{ then Adj } A =$$

$$\begin{bmatrix} A_{11} & A_{21} & \cdots & A_{1n} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}$$

It is easy to see that $A(\text{adj}A) = (\text{adj}A)A = |A|.I_n$.

Some More Special Type of Matrices

(i) Symmetric Matrix

A matrix $A = [a_{ij}]$ n×n is symmetric if A = A'. Note that aij = aji for such a matrix, ∀ 1 ≤ i, j ≤ n.

(ii) Skew Symmetric Matrix

A matrix A = [aij]n×n is skew symmetric if A = −A'. Note that $a_{ij} = -a_{ji}$ for such a matrix, ∀ 1 ≤ i, i ≤ n. If i = j, then aii = $-a_{ii} \Rightarrow a_{ii} = 0$. Thus in a skew symmetric matrix diagonal entries are zeros.

Inverse of a Square Matrix

Let A be any n – rowed square matrix. Then a matrix B, if exists, such that AB = BA = In, is called the inverse of A. Inverse of A is usually denoted by A–1 (if exists). We have

$|A|\, I_n = A(\mathrm{adj}_A) \Rightarrow |A|\, A^{-1} = (\mathrm{adj}_A)$. Thus the necessary and sufficient condition for a square matrix A to possess the inverse is that

$|A| \neq 0$ and then $A^{-1} = \dfrac{Adj(A)}{|A|}$. A square matrix A is called non–singular if $|A| \neq 0$. Hence a square matrix A is invertible if and only if A is non–singular.

System of Linear Simultaneous Equations

System of three linear equations with three unknowns is $a_1x + b_1y + c_1z = d_1$, $a_2x + b_2y + c_2z = d_2$, $a_3x + b_3y + c_3z = d_3$

- If system of equations has no solution, then it is called inconsistent.
- If system of equations has at least one solution, then it is called consistent.
- If $(d_1, d_2, d_2) = (0, 0, 0)$, then the system is called homogenous, otherwise non–homogenous.

System of Linear Non Homogenous Simultaneous Equations

Consider the system of linear non–homogenous simultaneous equations in three unknowns x, y and z, given by $a_1x + b_1y + c_1z = d_1$, $a_2x + b2y + c_2z = d_2$ and $a_3x + b_3y + c_3z = d_3$.

where $(d_1, d_2, d_3) \neq (0, 0, 0)$.

Let

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix},$$

$$\Delta = \begin{vmatrix} a_1 & b_2 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = |A|, \Delta_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

obtained on replacing first column of Δ by B.

Similarly $\Delta_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$ and $\Delta_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

It can be shown that AX = B, x.Δ = Δx, y.Δ = Δy and z.Δ = Δz.

Definition of a Determinant

A determinant is an arrangement of $n \times n$ numbers into n – horizontal lines (each line is called a row) and n vertical lines (each line is called a column) enclosed into two vertical bars. It is denoted by D or Δ.

A determinant having $n \times n$ numbers is called determinant of order n.

The number aij which appears in the ith row and jth column is called (ij)th entry of the determinant.

Diagonal Elements

The entries $a_{11}, a_{22}, \dots$, ann are called diagon entries and all other entries are called non-diagonal entries.

Minor

To each entry of a determinant we associat a number (uniquely determined) called its minor. The minor of aij, denoted by M_{ij}, is th determinant, which is obtained on deleting ith row and jth column from the original determinant.

(a) Determinant Method of Solution

(i) If $\Delta \neq 0$, then the given system of equations has unique solution, given by

$x = \Delta x/\Delta$, $y = \Delta y/\Delta$ and $z = \Delta z/\Delta$.

(ii) If $\Delta = 0$, then two sub cases arise:

(a) at least one of Δx, Δy and Δz is non-zero, say $\Delta x \neq 0$. Now in $x.\Delta = \Delta x$, L.H.S. is zero and R.H.S. is not equal to zero. Thus we have no value of x satisfying

$x.\Delta = \Delta x$. Hence given system of equations has no solution.

(β) $\Delta x = \Delta y = \Delta z = 0$. In the case the given equations are dependent. Delete one or tw equation from the given system (as the cas may be) to obtain independent equation(s The remaining equation(s) may have no solution or infinitely many solutions. For example in $x + y + z = 3$, $2x + 2y + 2z = 9$, $3x$ $3y + 3z = 12$,

$\Delta = \Delta x = \Delta y = \Delta z = 0$ and hence equations are dependent (infact third equation is the sum of first two equations). Now after deleting the third equation we obtain independent equations $x + y + z = 3$, $2x + 2y + 2z = 9$, which obviously have no solution (infact these are parallel planes) where as in $x + y + z = 3$, $2x - y + 3z = 4$, $3x + 4z = 7$, $\Delta = \Delta x = \Delta y = \Delta z = 0$ and hence equations are dependent (infact third equation is the sum of first two equations). Now after deleting any equation (say third) we obtain independent equations $x + y + z = 3$, $2x - y + 3z = 4$, which have infinitely many solutions (infact these are non parallel planes). For let $z = k \in R$, then $x = \frac{7-4k}{3}$ and $y = \frac{k+2}{3}$. Hence we get infinitely many solutions.

(b) Matrix Method of Solution

(i) If $\Delta \neq 0$, then A−1 exists and hence AX = B
A−1(AX) = A−1B $\Rightarrow$ X = A−1B and therefore unique values of x, y and z are obtained.

(ii) If $\Delta = 0$, then we from the matrix [A : B], known as augmented matrix (a matrix of order 3 × 4). Using row operations obtain a matrix from [A : B] whose last row

Cofactor

Again to each entry of a determinant we associate a number (uniquely determined) called its cofactor. The cofactor of a_{ij}, denoted by C_{ij}, is defined by

$$C_{ij} = (-1)^{i+j} M_{ij}$$

Value of a Determinant

$$\Delta = \begin{vmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ \vdots & \vdots & \\ a_{i1} & a_{i2} \cdots & a_{in} \\ \vdots & \vdots & \\ a_{n1} & a_{n2} \cdots & a_{nn} \end{vmatrix} = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in} \text{ for}$$

each i.

Also Δ =

$$(-1)^{i+1}a_{i1}M_{i1} + (-1)^{i+2}a_{i2}M_{i2} + \ldots + (-1)^{i+n}a_{in}M_{in}.$$

Note: $a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in}$ is called expansion of Δ w.r.t its ith row. Expansion w.r.t each row gives the same value and is called value of Δ.

Application 2 Find the value of $\begin{vmatrix} 2 & 3 & 1 \\ 5 & 4 & 7 \\ 1 & 0 & 2 \end{vmatrix}$..

Solution $M_{11} = \begin{vmatrix} 4 & 7 \\ 0 & 2 \end{vmatrix} = 8$, $M_{12} = \begin{vmatrix} 5 & 7 \\ 1 & 2 \end{vmatrix} = 3$ and M_{1}

$= \begin{vmatrix} 5 & 4 \\ 1 & 0 \end{vmatrix} = -4.$

$$\therefore \begin{vmatrix} 2 & 3 & 1 \\ 5 & 4 & 7 \\ 1 & 0 & 2 \end{vmatrix} = 2M_{11} - 3M_{12} + M_{13} = 2 \times 8 - 3 \times 3 +$$

$(-4) = 3$

Properties of Determinants

(i) A determinant formed by changing rows into columns and the columns into rows is called as transpose of a determinant and is represented by DT. The value of D = DT.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \text{ e.g. } \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{vmatrix}$$

(ii) If all the elements of a row (or column) are zero, then the determinant is zero.

$$\begin{vmatrix} 0 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ 0 & b_3 & c_3 \end{vmatrix} = 0, \begin{vmatrix} a_1 & b_1 & c_1 \\ 0 & 0 & 0 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

(iii) If any two rows or any two columns of a determinant are identical, then the determinant is zero.

$$\begin{vmatrix} a_1 & a_1 & c_1 \\ a_2 & a_2 & c_2 \\ a_3 & a_3 & c_3 \end{vmatrix} = 0, \text{ e.g. } \begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} = 0,$$ (Here 1st and IInd rows are identical)

(iv) The interchange of any two rows (columns) of the determinant results in change of it's sign.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = - \begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix} \text{ e.g.}$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} \overset{R_1 \Leftrightarrow R_2}{=} (-1) \begin{vmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{vmatrix}$$

(v) If all the elements of a row (column) of a determinant are multiplied by a non-zero constant, then the determinant gets multiplied by that constant.

$$\begin{vmatrix} a_1 & kb_1 & c_1 \\ a_2 & kb_2 & c_2 \\ a_3 & kb_3 & c_3 \end{vmatrix} = k \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ and k}$$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\text{e.g. } 2 \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 4 & 5 \end{vmatrix} = \begin{vmatrix} 2 & 4 & 6 \\ 2 & 3 & 1 \\ 3 & 4 & 5 \end{vmatrix}$$

(vi) A determinant remains unaltered under a column operation of the form Ci + αCj + βCk (j, k ≠ i) or a row operation of the form Ri + αRj + βRk(j, k ≠ i).

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 + 2a_1 + 3c_1 & c_1 \\ a_2 & b_2 + 2a_2 + 3c_2 & c_2 \\ a_3 & b_3 + 2a_3 + 3c_3 & c_3 \end{vmatrix}$$

e.g.

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} \xrightarrow{R_2 \to R_1 + \lambda R_2} \begin{vmatrix} 1+4\lambda & 2+5\lambda & 3+6\lambda \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

(vii) If each element of a row (column) of a determinant is a sum of two terms, then determinant can be written as sum of two determinant in following way

$$\begin{vmatrix} a_1 & b_1 & c_1 + d_1 \\ a_2 & b_2 & c_2 + d_2 \\ a_3 & b_3 & c_3 + d_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

(viii) Summation of determinants

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_4 & b_4 & c_4 \end{vmatrix} =$$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 + a_4 & b_3 + b_4 & c_3 + c_4 \end{vmatrix}$$

And $$\sum_{r=1}^{n} \begin{vmatrix} f(r) & g(r) & h(r) \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} =$$

$$\begin{vmatrix} \sum_{r=1}^{n} f(r) & \sum_{r=1}^{n} g(r) & \sum_{r=1}^{n} h(r) \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

(ix) Product of two determinants

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix} =$$

$$\begin{vmatrix} a_1 l_1 + b_1 m_1 + c_1 n_1 & a_1 l_2 + b_1 m_2 + c_1 n_2 & a_1 l_3 + b_1 m_3 + c_1 \\ a_2 l_1 + b_2 m_1 + c_2 n_1 & a_2 l_2 + b_2 m_2 + c_2 n_2 & a_2 l_3 + b_2 m_3 + c \\ a_3 l_1 + b_3 m_1 + c_3 n_1 & a_3 l_2 + b_3 m_2 + c_3 n_2 & a_3 l_3 + b_3 m_3 + \end{vmatrix}$$

(ix) Product of two determinants

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix} =$$

$$\begin{vmatrix} a_1l_1 + b_1m_1 + c_1n_1 & a_1l_2 + b_1m_2 + c_1n_2 & a_1l_3 + b_1m_3 + c_1n_3 \\ a_2l_1 + b_2m_1 + c_2n_1 & a_2l_2 + b_2m_2 + c_2n_2 & a_2l_3 + b_2m_3 + c_2n_3 \\ a_3l_1 + b_3m_1 + c_3n_1 & a_3l_2 + b_3m_2 + c_3n_2 & a_3l_3 + b_3m_3 + c_3n_3 \end{vmatrix}$$

=

$$\begin{vmatrix} a_1l_1 + b_1l_2 + c_1l_3 & a_1m_1 + b_1m_2 + c_1m_3 & a_1n_1 + b_1n_2 + c_1n_3 \\ a_2l_1 + b_2l_2 + c_2l_3 & a_2m_1 + b_2m_2 + c_2m_3 & a_2n_1 + b_2n_2 + c_2n_3 \\ a_3l_1 + b_3l_2 + c_3l_3 & a_3m_1 + b_3m_2 + c_3m_3 & a_3n_1 + b_3n_2 + c_3n_3 \end{vmatrix}$$

Remark:

Let $\Delta \neq 0$ and Δc denotes the determinant of co-factors then $\Delta c = \Delta n - 1$, where $n \in N$ is the order of determinant Δ.

TRIGONOMETRY

$\sin\theta = \frac{P}{H} = \frac{AB}{AC}$

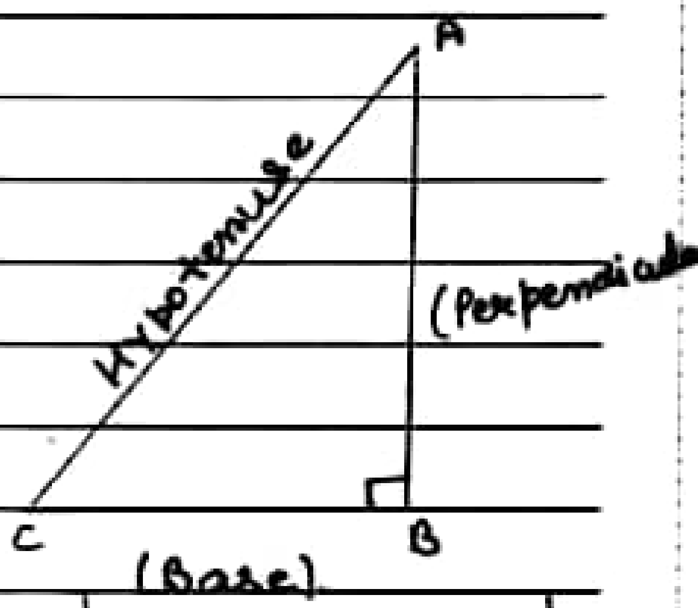
$\cos\theta = \frac{B}{H} = \frac{BC}{AC}$

$\tan\theta = \frac{P}{B} = \frac{AB}{BC}$

$\sec\theta = \frac{H}{B} = \frac{AC}{BC}$

$\cot\theta = \frac{B}{P} = \frac{BC}{AB}$

$\text{cosec}\,\theta = \frac{H}{P} = \frac{AC}{AB}$



$\sin\theta = \frac{1}{\text{cosec}\,\theta}$	$\cos\theta = \frac{1}{\sec\theta}$	$\cot\theta = \frac{1}{\tan\theta}$
$\text{cosec}\,\theta = \frac{1}{\sin\theta}$	$\sec\theta = \frac{1}{\cos\theta}$	$\tan\theta = \frac{1}{\cot\theta}$

$\frac{\sin\theta}{\cos\theta} = \tan\theta$

$\frac{\cos\theta}{\sin\theta} = \cot\theta$

$\sin\theta = \tan\theta \cdot \cos\theta$

$\cos\theta = \cot\theta \cdot \sin\theta$

1 Right angle = 90°

$1^\circ = 60'$ (minutes)

$1' = 60''$ (Seconds)

π radians = 180°

$1^c = \frac{180^\circ}{\pi}$

$1^\circ = \frac{\pi^c}{180}$

$\sin(90-\theta) = \cos\theta$

$\cos(90-\theta) = \sin\theta$

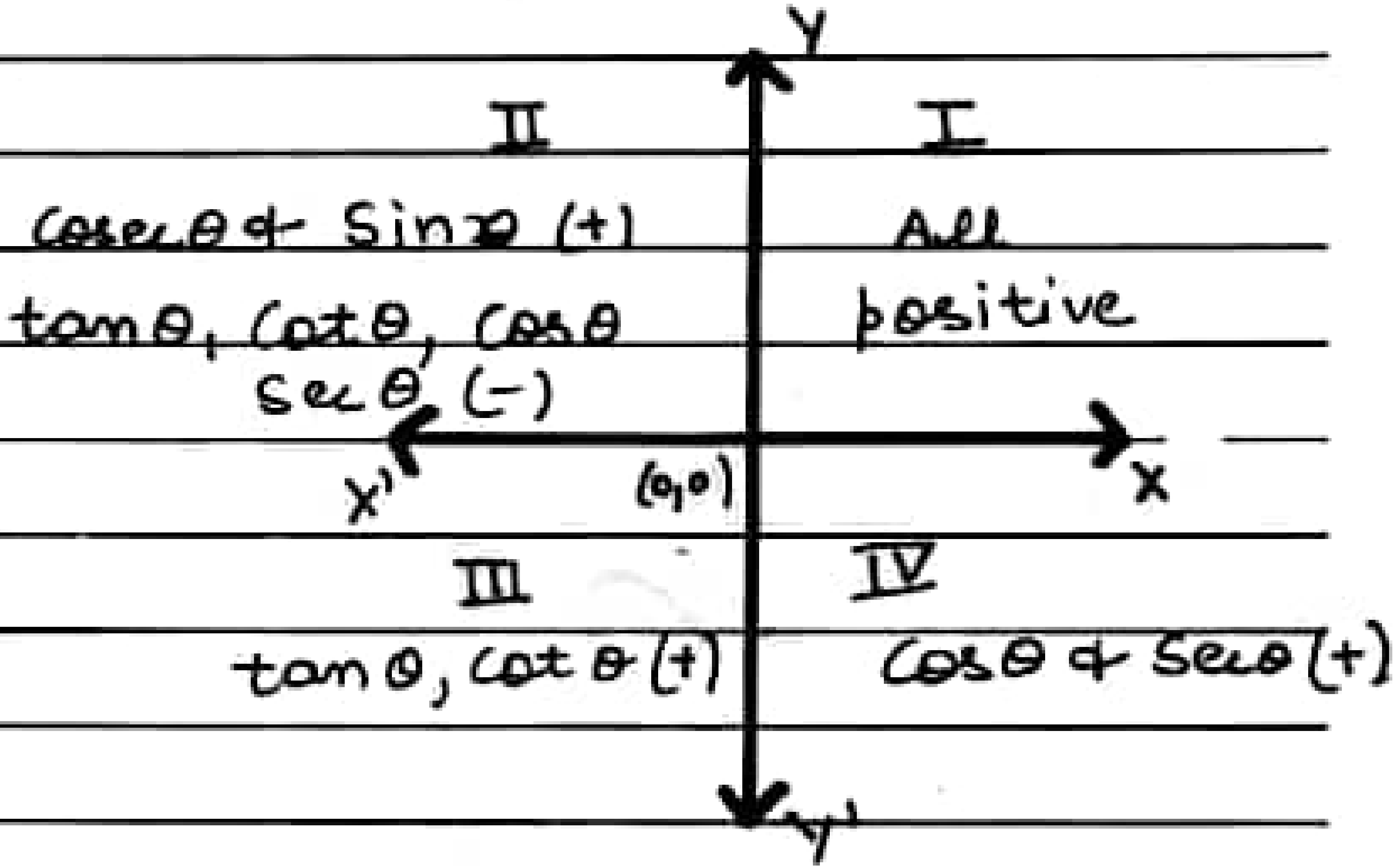
$\tan(90-\theta) = \cot\theta$

$\cot(90-\theta) = \tan\theta$

$\sec(90-\theta) = \text{cosec}\,\theta$

$\text{cosec}(90-\theta) = \sec\theta$

TRIGONOMETRY



$\sin(2\pi - x) = -\sin x$	$\sin(2\pi + x) = \sin x$
$\cos(2\pi - x) = \cos x$	$\cos(2\pi + x) = \cos x$
$\tan(2\pi - x) = -\tan x$	$\tan(2\pi + x) = \tan x$
$\sec(2\pi - x) = \sec x$	$\sec(2\pi + x) = \sec x$
$\cot(2\pi - x) = -\cot x$	$\cot(2\pi + x) = \cot x$
$\operatorname{cosec}(2\pi - x) = -\operatorname{cosec} x$	$\operatorname{cosec}(2\pi + x) = \operatorname{cosec} x$

→ $\cos(A-B) = \cos A \cos B + \sin A \sin B$

→ $\cos(A+B) = \cos A \cos B - \sin A \sin B$

→ $\sin(A-B) = \sin A \cos B - \cos A \sin B$

→ $\sin(A+B) = \sin A \cos B + \cos A \sin B$

→ $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \cdot \tan B}$

→ $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \cdot \tan B}$

→ $\cot(A+B) = \dfrac{\cot A \cdot \cot B - 1}{\cot B + \cot A}$

→ $\cot(A-B) = \dfrac{\cot A \cdot \cot B + 1}{\cot B - \cot A}$

$\sin(-x) = -\sin x$

$\cos(-x) = \cos x$

$\tan(-x) = -\tan x$

$\sec(-x) = \sec x$

$\cot(-x) = -\cot x$

$\operatorname{cosec}(-x) = -\operatorname{cosec} x$

$\sin(\pi/2 - x) = \cos x$

$\cos(\pi/2 - x) = \sin x$

$\tan(\pi/2 - x) = \cot x$

$\sec(\pi/2 - x) = \operatorname{cosec} x.$

$\cot(\pi/2 - x) = \tan x.$

$\operatorname{cosec}(\pi/2 - x) = \sec x.$

$\sin(\pi - x) = \sin x$

$\cos(\pi - x) = -\cos x$

$\tan(\pi - x) = -\tan x$

$\sec(\pi - x) = -\sec x$

$\cot(\pi - x) = -\cot x$

$\operatorname{cosec}(\pi - x) = \operatorname{cosec} x.$

$\sin(\pi/2 + x) = \cos x$

$\cos(\pi/2 + x) = -\sin x$

$\tan(\pi/2 + x) = -\cot x$

$\sec(\pi/2 + x) = -\operatorname{cosec} x$

$\cot(\pi/2 + x) = -\tan x$

$\operatorname{cosec}(\pi/2 + x) = \sec x$

$\sin(\pi + x) = -\sin x$

$\cos(\pi + x) = -\cos x$

$\tan(\pi + x) = \tan x$

$\sec(\pi + x) = -\sec x$

$\cot(\pi + x) = \cot x$

$\operatorname{cosec}(\pi + x) = -\operatorname{cosec} x$

⇒ Even function

If $f(-x) = f(x)$

⇒ Odd function

If $f(-x) = -f(x)$

$$\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$$

$$\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$$

$$\sin(A+B+C) = \sin A \cos B . \cos C + \cos A . \sin B \cos C + \cos A . \cos B \sin C - \sin A . \sin B \sin C$$

$$\cos(A+B+C) = \cos A \cos B \cos C - \cos A \sin B \sin C - \sin A \cos B \sin C - \sin A \sin B \cos C$$

$$\tan(A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$$

$$2 \sin A \cos B = \sin(A+B) + \sin(A-B)$$

$$2 \sin B \cos A = \sin(A+B) - \sin(A-B)$$

$$2 \cos A \cos B = \cos(A+B) + \cos(A-B)$$

$$2 \sin A \sin B = \cos(A-B) - \cos(A+B)$$

$$\sin C + \sin D = 2 \sin\left(\frac{C+D}{2}\right) \cos\left(\frac{C-D}{2}\right)$$

$$\sin C - \sin D = 2 \sin\left(\frac{C-D}{2}\right) \cos\left(\frac{C+D}{2}\right)$$

$$\cos C + \cos D = 2 \cos\left(\frac{C+D}{2}\right) \cos\left(\frac{C-D}{2}\right)$$

$$\cos D - \cos C = 2 \sin\left(\frac{C+D}{2}\right) \sin\left(\frac{C-D}{2}\right)$$

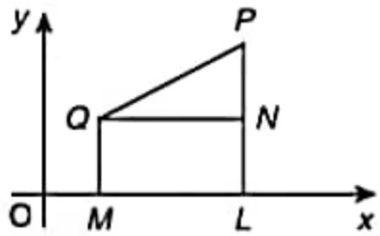
The method of representing a point by means of coordinates was first introduced by Rena Descartes and hence this branch of mathematics is called the rectangular Cartesian coordinate system.

Using this coordinate system, one can easily find the distance between two points in a plane, the coordinates of the point that divides a line segment in a given ratio, the centroid of a triangle, the area of a triangle and the locus of a point that moves according to a given geometrical law.

1.1.1 Distance between Two Given Points

Let P and Q be two points with coordinates (x_1, y_1) and (x_2, y_2).

Draw PL and QM perpendiculars to the x-axis, and draw QN perpendicular to PL. Then,



$$OL = x_1,\ LP = y_1,\ OM = x_2,\ MQ = y_2$$

$$QN = ML = OL - OM = x_1 - x_2$$

$$NP = LP - LN = LP - MQ = y_1 - y_2$$

In ΔPQN, $PQ^2 = QN^2 + NP^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$

$$\therefore\ PQ = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Note 1.1.1: The distance of P from the origin O is $OP = \sqrt{x_1^2 + y_1^2}$

Example 1.1.1

If P is the point (4, 7) and Q is (2, 3), then

$$PQ = \sqrt{(4-2)^2 + (7-3)^2} = \sqrt{4+16} = \sqrt{20} = 2\sqrt{5} \text{ units}$$

Example 1.1.2

The distance between the points $P(2, -5)$ and $Q(-4, 7)$ is

$$PQ = \sqrt{(2+4)^2 + (-5-7)^2} = \sqrt{36+144} = \sqrt{180} = 6\sqrt{5} \text{ units}$$

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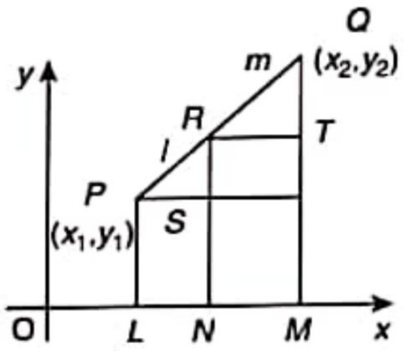
1.2 SECTION FORMULA

1.2.1 Coordinates of the Point that Divides the Line Joining Two Given Points in a Given Ratio

Let the two given points be $P(x_1, y_1)$ and $Q(x_2, y_2)$.



Let the point R divide PQ internally in the ratio $l:m$. Draw PL, QM and RN perpendiculars to the x-axis. Draw PS perpendicular to RN and RT perpendicular to MQ. Let the coordinates of R be (x, y). R divides PQ internally in the ratio $l:m$. Then,

$$OL = x_1,\ LP = y_1,\ OM = x_2,\ MQ = y_2,\ ON = x,\ NR = y$$

$$PS = LN = ON - OL = x - x_1;\ RT = NM = OM - ON = x_2 - x$$

$$SR = NR - NS = NR - LP = y - y_1;\ TQ = MQ - MT = y_2 - y_1$$

Triangles PSR and RTQ are similar.

$$\therefore \quad \frac{PS}{RT} = \frac{SR}{TQ} = \frac{PR}{RQ} = \frac{l}{m}$$

$$\frac{PS}{RT} = \frac{l}{m} \Rightarrow \frac{x - x_1}{x_2 - x} = \frac{l}{m} \Rightarrow m(x - x_1) = l(x_2 - x)$$

$$\therefore x(l + m) = lx_2 + mx_1 \qquad \text{or } x = \frac{lx_2 + mx_1}{l + m}$$

Also

$$\frac{SR}{TQ} = \frac{l}{m} \Rightarrow \frac{y - y_1}{y_2 - y} = \frac{l}{m} \Rightarrow y = \frac{ly_2 + my_1}{l + m}$$

Hence, the coordinates of R are $\left(\dfrac{lx_2 + mx_1}{l + m}, \dfrac{ly_2 + my_1}{l + m}\right)$.

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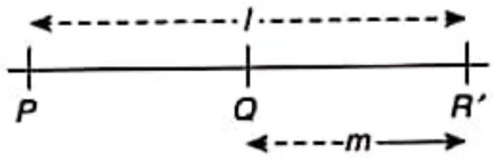
1.2.2 External Point of Division



If the point R' divides PQ externally in the ratio $l{:}m$, then

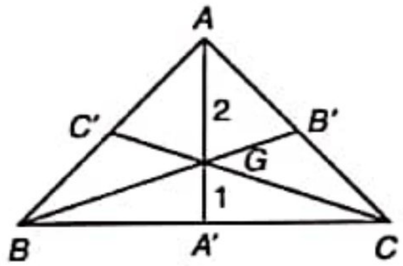
$$\frac{PR'}{R'Q} = \frac{l}{m} \Rightarrow \frac{PR'}{QR'} = \frac{l}{-m}$$

Choosing m negative, we get the coordinates of R'. Therefore, the coordinates of R' are $\left(\frac{lx_2 - mx_1}{l-m}, \frac{ly_2 - my}{l-m}\right)$.

Note 1.2.2.1: If we take $l = m = 1$ in the internal point of division, we get the coordinates of the midpoint. Therefore, the coordinates of the midpoint of PQ are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

1.2.3 Centroid of a Triangle Given its Vertices

Let ABC be a triangle with vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$.



Let AA', BB' and CC' be the medians of the triangle. Then A', B', C' are the midpoints of the sides BC, CA and AB, respectively. The coordinates of A' are $\left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}\right)$. We know that the medians of a triangle are concurrent at the point G called the centroid and G divides each median in the ratio 2:1. Considering the median AA', the coordinates of G are

$$\left(\frac{1 \cdot x_1 + 2 \cdot \left(\frac{x_2 + x_3}{2}\right)}{1+2}, \frac{1 \cdot y_1 + 2 \cdot \left(\frac{y_2 + y_3}{2}\right)}{1+2}\right)$$

(i.e.) $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$

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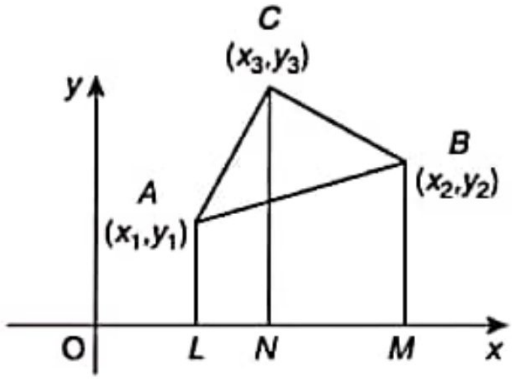
1.2.4 Area of Triangle *ABC* with Vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$

Let the vertices of triangle *ABC* be $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$.



Draw *AL*, *BM* and *CN* perpendiculars to *OX*. Then, area Δ of triangle *ABC* is calculated as

Δ = Area of trapezium *ALNC* + Area of trapezium *CNMB* − Area of trapezium *ALMB*

$$= \frac{1}{2}(LA + NC)\cdot LN + \frac{1}{2}(NC + MB)\cdot NM - \frac{1}{2}(LA + MB)\cdot LM$$

$$= \frac{1}{2}(y_1 + y_3)(x_3 - x_1) + \frac{1}{2}(y_3 + y_2)(x_2 - x_3) - \frac{1}{2}(y_1 + y_2)(x_2 - x_1)$$

$$= \frac{1}{2}[x_1(y_1 + y_2 - y_1 - y_3) + x_2(y_3 + y_2 - y_1 - y_2) + x_3(y_1 + y_3 - y_3 - y_2)]$$

$$= \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$$

$$= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Note 1.2.4.1: The area is positive or negative depending upon the order in which we take the points. Since scalar area is always taken to be a positive quantity, we take

$$\Delta = \frac{1}{2}\left|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right|$$

Note 1.2.4.2: If the vertices of the triangle are (0, 0), (x_1, y_1) and (x_2, y_2), then $\Delta = \frac{1}{2}\left|x_1 y_2 - x_2 y_1\right|$.

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distance from two fixed points are equal, then the locus of the point is the perpendicular bisector of the line joining the two fixed points. If A and B are two fixed points and point P moves such that $\angle APB = \frac{\pi}{2}$ then the locus of P is a circle with AB as the diameter. It is possible to represent the locus of a point by means of an equation.

Suppose a point $P(x, y)$ moves such that its distance from two fixed points $A(2, 3)$ and $B(5, -3)$ are equal. Then the geometrical law is $PA = PB \Rightarrow PA^2 = PB^2$

$$\Rightarrow (x-2)^2 + (y-3)^2 = (x-5)^2 + (y+3)^2$$
$$\Rightarrow x^2 - 4x + 4 + y^2 + 9 - 6y = x^2 + 25 - 10x + y^2 + 9 + 6y$$
$$\Rightarrow 6x - 12y - 21 = 0 \quad \Rightarrow 2x - 4y - 7 = 0$$

Here, the locus of P is a straight line.

ILLUSTRATIVE EXAMPLES

Example 1.1

Find the distance between the points (4, 7) and (–2, 5).

Solution

Let P and Q be the points (4, 7) and (–2, 5), respectively.

$$PQ^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 = (4+2)^2 + (7-5)^2 = 36 + 4 = 40$$
$$PQ = \sqrt{40} \text{ units}$$

Example 1.2

Prove that the points (4, 3), (7, –1) and (9, 3) are the vertices of an isosceles triangle.

Solution

Let $A(4, 3)$, $B(7, -1)$, $C(9, 3)$ be the three given points.

Then $AB^2 = (4-7)^2 + (3+1)^2 = 9 + 16 = 25$

$$BC^2 = (7-9)^2 + (-1-3)^2 = 4 + 16 = 20$$
$$AC^2 = (4-9)^2 + (3-3)^2 = 25 + 0 = 25$$
$$\therefore AB = 5, BC = \sqrt{20}, AC = 5$$

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Since the sum of two sides is greater than the third, the points form a triangle. Moreover, $AB = AC = 5$. Therefore, the triangle is an isosceles triangle.

Example 1.3

Show that the points (6, 6), (2, 3) and (4, 7) are the vertices of a right angled triangle.

Solution

Let A, B, C be the points (6, 6), (2, 3) and (4, 7), respectively.

$$AB^2 = (6-2)^2 + (6-3)^2 = 16 + 9 = 25$$

$$BC^2 = (2-4)^2 + (3-7)^2 = 4 + 16 = 20$$

$$AC^2 = (6-4)^2 + (6-7)^2 = 4 + 1 = 5$$

$$AC^2 + BC^2 = 20 + 5 = 25 = AB^2$$

Hence, the points are the vertices of a right angled triangle.

Example 1.4

Show that the points (7, 9), (3, −7) and (−3, 3) are the vertices of a right angled isosceles triangle.

Solution

Let A, B, C be the points (7, 9), (3, −7), (−3, 3), respectively.

$$AB^2 = (7-3)^2 + (9+7)^2 = 16 + 256 = 272$$

$$BC^2 = (3+3)^2 + (-7-3)^2 = 36 + 100 = 136$$

$$AC^2 = (7+3)^2 + (9-3)^2 = 100 + 36 = 136$$

$$BC^2 + AC^2 = 136 + 136 = 272 = AB^2$$

Hence, the points are vertices of a right angled triangle. Also, $BC = AC$. Therefore, it is a right angled isosceles triangle.

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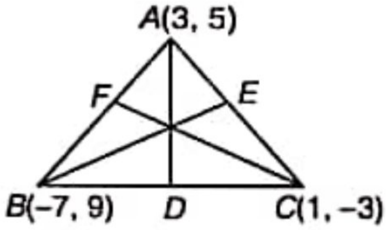
Example 1.26

The vertices of a triangle are $A(3, 5)$, $B(-7, 9)$ and $C(1, -3)$. Find the length of the three medians of the triangle.

Solution



Let D, E and F be the midpoints of the sides of BC, CA and AB, respectively. The coordinates of D are $\left(\frac{-7+1}{2}, \frac{9-3}{2}\right)$ (i.e.) $(-3, 3)$. The coordinates of E are $\left(\frac{3+1}{2}, \frac{5-3}{2}\right)$ (i.e.) $(2, 1)$. The coordinates of F are $\left(\frac{3-7}{2}, \frac{5+9}{2}\right)$ (i.e.) $(-2, 7)$.

$$AD^2 = (3+3)^2 + (5-3)^2 = 36 + 4 = 40$$
$$BE^2 = (-7-2)^2 + (9-1)^2 = 81 + 64 = 145$$
$$CF^2 = (1+2)^2 + (-3-7)^2 = 9 + 100 = 109$$

Hence, the lengths of the medians are $AD = 2\sqrt{10}$, $BE = \sqrt{145}$ and $CF = \sqrt{109}$ units.

Example 1.27

Two of the vertices of a triangle are $(4, 7)$ and $(-1, 2)$ and the centroid is at the origin. Find the third vertex.

Solution

Let the third vertex of the triangle be (x, y). Then

$$\frac{4-1+x}{3} = 0 \text{ and } \frac{7+2+y}{3} = 0 \quad \therefore x = -3,\ y = -9.$$

Hence, the third vertex is $(-3, -9)$.

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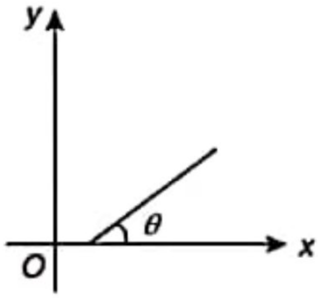
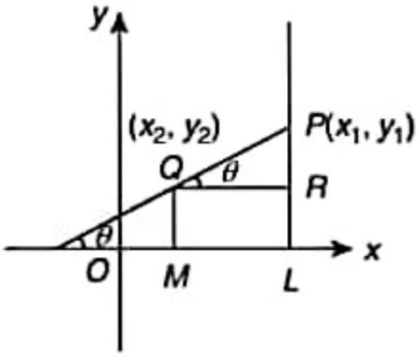
2.2 SLOPE OF A STRAIGHT LINE

If a straight line makes an angle θ with the positive direction of x-axis then $\tan\theta$ is called the slope of the straight line and is denoted by m.

$$\therefore m = \tan\theta.$$



We can now determine the slope of a straight line in terms of coordinates of two points on the line. Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be the two given points on a line. Draw PL and QM perpendiculars to x-axis. Let PQ make an angle θ with OX.



Draw QR perpendicular to LP. Then $\angle RQP = \theta$.

$$\therefore \quad QR = ML = OL - OM = x_1 - x_2.$$

$$RP = LR - LP = LP - QM = y_1 - y_2.$$

In ΔPQR, $\tan\theta = \dfrac{RP}{QR}$ (i.e.) $m = \dfrac{y_1 - y_2}{x_1 - x_2}$

2.3 SLOPE-INTERCEPT FORM OF A STRAIGHT LINE

Find the equation of the straight line, which makes an angle θ with OX and cuts off an intercept c on the y-axis.

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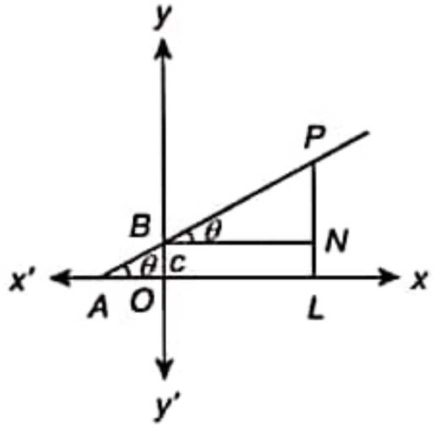


Let $P(x, y)$ be any point on the straight line which makes an angle θ with x-axis.

$\underline{|OAP} = \theta$, $OB = c = y$-intercept. Draw PL perpendicular to x-axis and BN perpendicular to LP. Then, $\underline{|NBP} = \theta$. $BN = OL = x$.

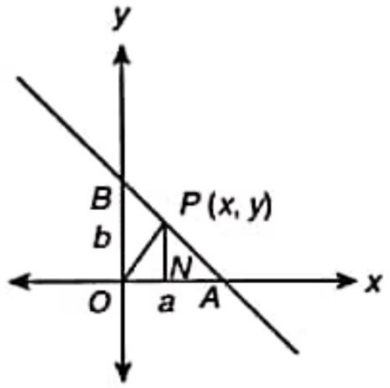
$$\therefore \quad NP = LP - LN = LP - OB = y - c.$$

In $\Delta NBP, \tan\theta = \dfrac{NP}{BN}$ (i.e.) $m = \dfrac{y-c}{x} \Rightarrow y = mx + c$.

This equation is true for all positions of P on the straight line. Hence, this is the equation of the required line.

2.4 INTERCEPT FORM

Find the equation of the straight line, which cuts off intercepts a and b, respectively on x and y axes.



Let $P(x, y)$ be any point on the straight line which meets x and y axes at A and B, respectively. Let $OA = a$, $OB = b$, $ON = x$, and $NP = y$; $NA = OA - ON = a - x$. Triangles PNA and BOA are similar. Therefore, $\dfrac{PN}{OB} = \dfrac{NA}{OA}$ (i.e.) $\dfrac{y}{b} = \dfrac{a-x}{a} \Rightarrow \dfrac{y}{b} = 1 - \dfrac{x}{a}$ or $\dfrac{x}{a} + \dfrac{y}{b} = 1$. This result is true for

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all positions of P on the straight line and hence this is the equation of the required line.

2.5 SLOPE-POINT FORM

Find the equation of the straight line with slope m and passing through the given point (x_1, y_1).

The equation of the straight line with a given slope m is

$$y = mx + c \tag{2.4}$$

Here, c is unknown. This straight line passes through the point (x_1, y_1). The point has to satisfy the equation $y = mx + c$.

$\therefore y_1 = mx_1 + c$. Substituting the value of c in equation (2.4), we get the equation of the line as

$$y = mx + y_1 - mx_1 \Rightarrow y - y_1 = m(x - x_1).$$

2.6 TWO POINTS FORM

Find the equation of the straight line passing through two given points (x_1, y_1) and (x_2, y_2).

$$y - y_1 = m(x - x_1) \tag{2.5}$$

where, m is unknown. The slope of the straight line passing through the points

$$(x_1, y_1) \text{ and } (x_2, y_2) \text{ is } m = \frac{y_1 - y_2}{x_1 - x_2} \tag{2.6}$$

By substituting equation (2.6) in equation (2.5), we get the required straight line $y - y_1 = \frac{y_1 - y_2}{x_1 - x_2}(x - x_1)$.

$$\text{(i.e.)} \quad \frac{y - y_1}{x - x_1} = \frac{y_1 - y_2}{x_1 - x_2}.$$

2.7 NORMAL FORM

Find the equation of a straight line in terms of the perpendicular p from the origin to the line and the angle that the perpendicular line makes with axis.

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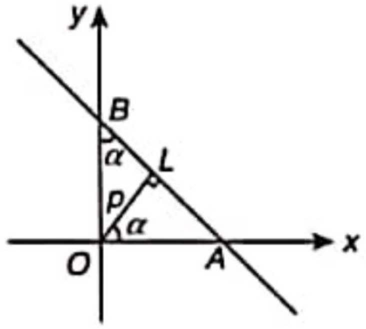


Draw $OL \perp AB$. Let $OL = p$.

Let $\lfloor AOL = \alpha$

$$\therefore \lfloor OBA = \alpha$$

$$\frac{OA}{OL} = \sec\alpha$$

$$OA = OL\sec\alpha = p\sec\alpha$$

$$\frac{OB}{OL} = \operatorname{cosec}\alpha$$

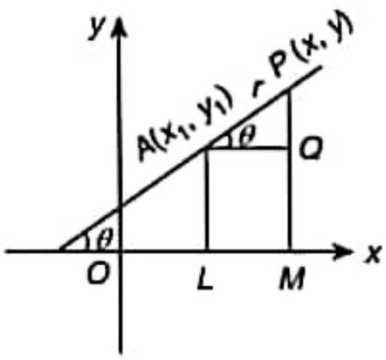
$$\therefore OB = OL\operatorname{cosec}\alpha = p\operatorname{cosec}\alpha.$$

Therefore, the equation of the straight line AB is $\dfrac{x}{p\sec\alpha} + \dfrac{y}{p\operatorname{cosec}\alpha} = 1.$

(i.e.) $\quad x\cos\alpha + y\sin\alpha = p$

2.8 PARAMETRIC FORM AND DISTANCE FORM

Let a straight line make an angle θ with x-axis and $A(x_1, y_1)$ be a point on the line. Draw AL, PM perpendicular to x-axis and AQ perpendicular to PM. Then, $\lfloor PAQ = \theta$.



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$$AQ = LM = OM - OL = x - x_1.$$

$$QP = MP - MQ = MP - LA = y - y_1. \text{ Let } AP = r.$$

In ΔPAQ, $x - x_1 = r\cos\theta$; $y - y_1 = r\sin\theta$.

$$\therefore \frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r.$$

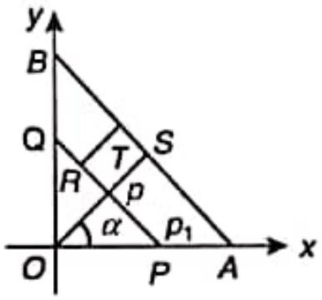
These are the parametric equations of the given line.

Note 2.8.1: Any point on the line is $x = x_1 + r\cos\theta$, $y = y_1 + r\sin\theta$.

Note 2.8.2: r is the distance of any point on the line from the given point $A(x_1, y_1)$.

2.9 PERPENDICULAR DISTANCE ON A STRAIGHT LINE

Find the perpendicular distance from a given point to the line $ax + by + c = 0$.



Let $R(x_1, y_1)$ be a given point and $ax + by + c = 0$ be the given line. Through R draw the line PQ parallel to AB. Draw OS perpendicular to AB meeting PQ at T. Let $OS = p$ and $PT = p_1$. Let $\angle AOS = \alpha$. Then the equation of AB is

$$x\cos\alpha + y\sin\alpha = p \tag{2.7}$$

which is the same as

$$ax + by = -c \tag{2.8}$$

Equations (2.7) and (2.8) represent the same line and, therefore, identifying we get $\dfrac{\cos\alpha}{a} = \dfrac{\sin\alpha}{b} = \dfrac{-p}{c} = \pm\dfrac{\sqrt{\cos^2\alpha + \sin^2\alpha}}{\sqrt{a^2 + b^2}}$

$$\therefore \cos\alpha = \pm\frac{a}{\sqrt{a^2 + b^2}}; \ \sin\alpha = \pm\frac{b}{\sqrt{a^2 + b^2}}.$$

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$$p = \mp \frac{c}{\sqrt{a^2 + b^2}}.$$

The equation of the line PQ is $x\cos\alpha + y\sin\alpha = p$. Since the point $R(x_1, y_1)$ lies on the line $x_1\cos\alpha + y_1\sin\alpha - p_1 = 0$.

$$\therefore p_1 = x_1\cos\alpha + y_1\sin\alpha.$$

Then, the length of the perpendicular line from R to AB

$$= p - p_1 = p - x_1\cos\alpha - y_1\sin\alpha = \mp\frac{c}{\sqrt{a^2+b^2}} \mp \frac{ax_1}{\sqrt{a^2+b^2}} \mp \frac{by_1}{\sqrt{a^2+b^2}}$$

$$= \mp\frac{ax_1 + by_1 + c}{\sqrt{a^2+b^2}} = \left|\frac{ax_1 + by_1 + c}{\sqrt{a^2+b^2}}\right|$$

Note 2.9.1: The perpendicular distance from the origin on the line $ax + by + c = 0$ is $\dfrac{|c|}{\sqrt{a^2+b^2}}$.

2.10 INTERSECTION OF TWO STRAIGHT LINES

Let the two intersecting straight lines be $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$. Let the straight lines intersect at the point (x_1, y_1). Then (x_1, y_1) lies on both the lines and hence satisfy these equations. Then

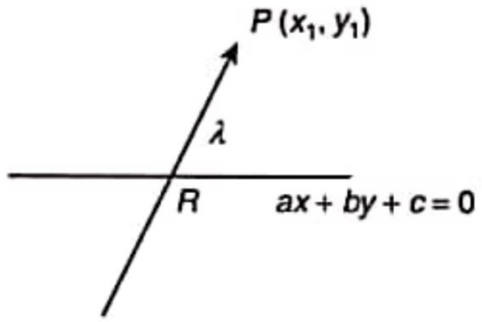
$$a_1x_1 + b_1y_1 + c_1 = 0 \quad \text{and} \quad a_2x_2 + b_2y_2 + c_2 = 0.$$

Solving the equations, we get

$$\frac{x_1}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1}$$

Therefore, the point of intersection is $\left(\dfrac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \dfrac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right)$.

Find the ratio at which the line $ax + by + c = 0$ divides the line joining the points (x_1, y_1) and (x_2, y_2).



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Let the line $ax + by = c = 0$ divide the line joining the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ in the ratio $\lambda{:}1$. Then, the coordinates of the point of division R are $\left(\frac{x_1+\lambda x_2}{1+\lambda}, \frac{y_1+\lambda y_2}{1+\lambda}\right)$.

This point lies on the line $ax + by + c = 0$.

$$\therefore a\left(\frac{x_1+\lambda x_2}{1+\lambda}\right)+b\left(\frac{y_1+\lambda y_2}{1+\lambda}\right)+c=0.$$

$$\text{(i.e.)}\quad a\,(x_1+\lambda x_2)+b\,(y_1+\lambda y_2)+c\,(1+\lambda)=0.$$

$$ax_1+by_1+c+\lambda\,(ax_2+by_2+c_2)=0.$$

$$\Rightarrow \lambda=-\frac{ax_1+by_1+c}{ax_2+by_2+c_2}.$$

Note 2.10.1:

1. If λ is positive then the points (x_1, y_1) and (x_2, y_2) lie on the opposite sides of the line $ax + by + c = 0$.
2. If λ is negative then the points (x_1, y_1) and (x_2, y_2) lie on the same side of the line $ax + by + c = 0$.
3. In other words, if the expressions $ax_1 + by_1 + c$ and $ax_2 + by_2 + c_2$ are of opposite signs then the point (x_1, y_1) and (x_2, y_2) lie on the opposite sides of the line $ax + by + c = 0$. If they are of the same sign then the points (x_1, y_1) and (x_2, y_2) lie on the same side of the line $ax + by + c = 0$.

Find the equation of a straight line passing through intersection of the lines $a_1x + b_1y + c = 0$ and $a_2x + b_2y + c = 0$.

Consider the equation $a_1x + b_1y + c_1 + l\,(a_2x + b_2y + c_2) = 0$. (2.9)

This is a linear equation in x and y and hence this equation represents a straight line. Let (x_1, y_1) be the point of intersection of the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$. Then (x_1, y_1) has to satisfy the two equations:

$$a_1x_1 + b_1y_1 + c_1 = 0 \qquad (2.10)$$

$$a_2x_2 + b_2y_2 + c_2 = 0 \qquad (2.11)$$

On multiplying equation (2.11) by λ and adding with equation (2.10) we get,

$$(a_1x_1 + b_1y_1 + c_1) + \lambda\,(a_2x_2 + b_2y_2 + c_2) = 0.$$

This equation shows that the point $x = x_1$ and $y = y_1$ satisfies equation (2.9). Hence the point (x_1, y_1) lies on the straight line given by the equation (2.9),

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which is a line passing through the intersection of the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$.

2.11 CONCURRENT STRAIGHT LINES

Consider three straight lines given by equations:

$$a_1x + b_1y + c_1 = 0 \quad (2.12)$$

$$a_2x + b_2y + c_2 = 0 \quad (2.13)$$

$$a_3x + b_3y + c_3 = 0 \quad (2.14)$$

The point of intersection of lines given by equations (2.12) and (2.13) is $\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \frac{c_1c_2 - c_2a_1}{a_1b_2 - a_2b_1}\right)$.

If the three given lines are concurrent, the above point should lie on the straight line given by equation (2.14).

$$\text{Then, } a_3\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}\right) + \left(\frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right)b_3 + c_3 = 0.$$

$$\text{(i.e.)} \quad a_3(b_1c_2 - b_2c_1) + b_3(c_1a_2 - c_2a_1) + c_3(a_1b_2 - a_2b_1) = 0.$$

This is the required condition for the three given lines to be concurrent. The above condition can be expressed in determinant form $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$

If l, m, and n are constants such that $l(a_1x + b_1y + c_1) + m(a_2x + b_2y + c_2) + n(a_3x + b_3y + c_3)$ vanishes identically then prove that the lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$, and $a_3x + b_3y + c_3 = 0$ are concurrent.

Let the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ meet at the point (x_1, y_1).

$$\text{Then} \quad a_1x_1 + b_1y_1 + c_1 = 0 \quad (2.15)$$

$$\text{and} \quad a_2x_2 + b_2y_2 + c_2 = 0 \quad (2.16)$$

For all values of x and y given that,

$$l(a_1x + b_1y + c_1) + m(a_2x + b_2y + c_2) + n(a_3x + b_3y + c_3) = 0 \quad (2.17)$$

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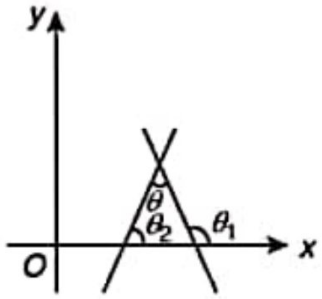
Then it will be true for $x = x_1$ and $y = y_1$.

$\therefore l(a_1x_1 + b_1y_1 + c_1) + m(a_2x_1 + b_2y_1 + c_2) + n(a_3x_1 + b_3y_1 + c_3) = 0.$ Using equations (2.15) and (2.16), we get $a_3x_1 + b_3y_1 + c_3 = 0$. That is, the point (x_1, y_1) lies on the line $a_3x + b_3y + c_3 = 0$.

Therefore, the lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$, $a_3x + b_3y + c_3 = 0$ are concurrent at (x_1, y_1).

2.12 ANGLE BETWEEN TWO STRAIGHT LINES



Let θ be the angle between two straight lines, whose slopes are m_1 and m_2. Let the two lines with slopes m_1 and m_2 make angles θ_1 and θ_2 with x-axis. Then, $m_1 = \tan\theta_1$, $m_2 = \tan\theta_2$. Also, $\theta = \theta_1 - \theta_2$

$$\tan\theta = \tan(\theta_1 - \theta_2) = \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2} = \frac{(m_1 - m_2)}{1 + m_1 m_2}.$$

If the RHS is positive, then θ is the acute angle between the lines. If RHS is negative, then θ is the obtuse angle between the lines.

$$\therefore \tan\theta = \pm\frac{(m_1 - m_2)}{1 + m_1 m_2} \Rightarrow \theta = \tan^{-1}\left(\pm\frac{(m_1 - m_2)}{1 + m_1 m_2}\right)$$

Note 2.12.1: If the lines are parallel then $\theta = 0$ and $\tan\theta = \tan 0 = 0$.

$$\therefore \frac{m_1 - m_2}{1 + m_1 m_2} = 0 \Rightarrow m_1 = m_2.$$

Note 2.12.2: If the lines are perpendicular then, $\theta = \frac{\pi}{2}$.

$$\tan\frac{\pi}{2} = \pm\frac{m_1 - m_2}{1 + m_1 m_2} \Rightarrow 1 + m_1 m_2 = 0 \Rightarrow m_1 m_2 = -1.$$

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Example 2.4

Find the slope of the line $2x - 3y + 7 = 0$.

Solution

The equation of the line is $2x - 3y + 7 = 0$ (i.e.) $3y = 2x + 7$.

$$\text{(i.e.)} \quad y = \frac{2}{3}x + \frac{7}{3}.$$

Therefore, slope of the line $= \frac{2}{3}$.

Example 2.5

Find the equation of the straight line making an angle 135° with the positive direction of x-axis and cutting of an intercept 5 on the y-axis.

Solution

The slope of the straight line is

$$\begin{aligned} m &= \tan\theta = \tan 135^\circ \\ &= \tan(180 - 45) \\ &= -\tan 45^\circ \\ &= -1. \end{aligned}$$

y intercept $= c = 5$. Therefore, the equation of the straight line is

$$y = mx + c \quad \text{(i.e.)} \quad y = +5 - x \text{ or } x + y = 5.$$

Example 2.6

Find the equation of the straight line cutting off the intercepts 2 and −5 on the axes.

Solution

The equation of the straight line is $\frac{x}{a} + \frac{y}{b} = 1$. Here, $a = 2$ and $b = -5$.

Therefore, the equation of the straight line is $\frac{x}{2} - \frac{y}{5} = 1$ or $5x - 2y = 10$.

Example 2.7

Find the equation of the straight line passing through the points (7, −3) and cutting off equal intercepts on the axes.

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Solution

Let the equation of the straight line be $\frac{x}{a}+\frac{y}{b}=1$.

$$\text{(i.e.)} \quad x+y=a.$$

This straight line passes through the point (7, –3).

Therefore, $7-3=a$ (i.e.) $a=4$.

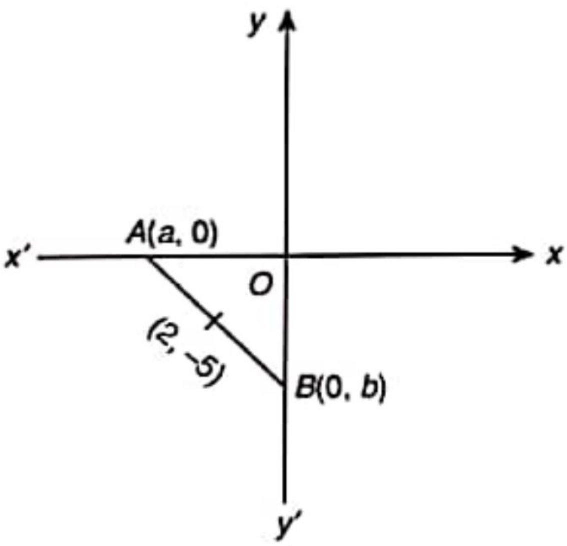
$\therefore$ The equation of the straight line is $x+y=4$.

Example 2.8

Find the equation of the straight line, the portion of which between the axes is bisected at the point (2, –5).

Solution

Let the equation of the straight line be $\frac{x}{a}+\frac{y}{b}=1$.



Let the line meet the x and y axes at A and B, respectively. Then the coordinates of A and B are $(a, 0)$ and $(0, b)$. The midpoint of AB is $\left(\frac{a}{2}, \frac{b}{2}\right)$. However, the midpoint is given as (2, –5).

Therefore, $\frac{a}{2}=2$ and $\frac{b}{2}=-5$.

$$\therefore a=4 \quad \text{and} \quad b=-10.$$

Hence, the equation of the straight line is $\frac{x}{4}-\frac{y}{10}=1$.

$$\text{(i.e.)} \quad 5x-2y=20.$$

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Example 2.15

Prove that the lines $3x - 4y + 5 = 0$, $7x - 8y + 5 = 0$, and $4x + 5y = 45$ are concurrent.

Solution

Given

$$3x - 4y = -5 \quad (2.22)$$
$$7x - 8y = -5 \quad (2.23)$$
$$4x + 5y = 45 \quad (2.24)$$

Solving equations (2.22) and (2.23), we get the point of intersection of the two lines.

$$(2.22)\times 2 \quad 6x - 8y = -10$$
$$(2.23)\times 1 \quad 7x - 8y = -5$$
$$x = 5$$

$\therefore$ From equation (2.22), $15 - 4y = -5$.

$\therefore y = 5$.

Hence, the point of intersection of the lines is (5, 5). Substituting $x = 5$ and $y = 5$, in equation (2.24), we get $20 + 25 = 45$ which is true.

$\therefore$ The third line also passes through the points (5, 5). Hence it is proved that the three lines are concurrent.

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Example 2.16

Find the value of a so that the lines $x - 6y + a = 0$, $2x + 3y + 4 = 0$, and $x + 4y + 1 = 0$ are concurrent.

Solution

Given
$$x - 6y + a = 0 \tag{2.25}$$
$$2x + 3y + 4 = 0 \tag{2.26}$$
$$x + 4y + 1 = 0 \tag{2.27}$$

Solving the equations (2.26) and (2.27) we get,

$$(2.26)\times 1 : 2x + 3y = -4$$
$$(2.27)\times 2 : \underline{2x + 8y = -2}$$

On subtracting, we get $5y = 2$

$$\therefore y = \frac{2}{5}$$

$$\therefore \text{ From equation (2.27) } x = \frac{-8}{5} - 1 = \frac{-8-5}{5}$$

$$\therefore x = \frac{-13}{5}$$

Hence, the point of intersection of the lines is $\left(\frac{-13}{5}, \frac{2}{5}\right)$. Since the lines are concurrent this point should lie on $x - 6y + a = 0$.

$$\text{(i.e.)} \quad \frac{-13}{5} - \frac{12}{5} + a = 0.$$
$$\therefore a = 5.$$

Example 2.17

Prove that for all values of λ the straight line $x(2 + 3\lambda) + y(3 - \lambda) - 5 - 2\lambda = 0$ passes through a fixed point. Find the coordinates of the fixed point.

Solution

$x(2 + 3\lambda) + y(3 - \lambda) - 5 - 2\lambda = 0$. This equation can be written in the form

$$2x + 3y - 5 + \lambda(3x - y - 2) = 0 \tag{2.28}$$

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This equation represents a straight line passing through the intersection of lines

$$2x + 3y - 5 = 0 \qquad (2.29)$$

$$3x - y - 2 = 0 \qquad (2.30)$$

for all values of λ.

(2.29) $\qquad 2x + 3y = 5$

(2.28) × 3 $\qquad 9x - 3y = 6$

On adding, we get $11x = 11 \Rightarrow x = 1$ and hence from equation (2.29) we get $y = 1$.

Therefore, the point of intersection of straight lines (2.29) and (2.30) is (1, 1). The straight line (2.28) passes through the point (1, 1) for all values of λ. Hence (2.28) passes through the fixed point (1, 1).

Example 2.18

Find the equation of the straight line passing through the intersection of the lines $3x - y = 5$ and $2x + 3y = 7$ and making an angle of 45° with the positive direction of x-axis.

Solution

Solving the equations, $3x - y = 5 \qquad (2.31)$

$$2x + 3y = 7 \qquad (2.32)$$

We get,

(2.31) × 3 $\qquad 9x - 3y = 15$

(2.32) $\qquad 2x + 3y = 7$

On adding, we get $11x = 22$.

$$\therefore x = 2.$$

From equation (2.31), $6 - y = 5$.

$$\therefore y = 1.$$

Hence (2, 1) is the point of intersection of the lines (2.31) and (2.32).

The slope of the required line is $m = \tan\theta$, $m = \tan 45^\circ = 1$. Therefore, the equation of the required line is $y - y_1 = m(x - x_1)$ (i.e.) $y - 1 = 1(x - 2) \Rightarrow x - y = 1$.

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Example 2.19

Find the equation of the straight line passing through the intersection of the lines $7x + 3y = 7$ and $2x + y = 2$ and cutting off equal intercepts on the axes.

Solution

The point of intersection of the lines is obtained by solving the following two equations:

$$7x + 3y = 7 \tag{2.33}$$

$$2x + y = 2 \tag{2.34}$$

(2.33) $\qquad 7x + 3y = 7$

(2.34) $\times 3$ $\qquad 6x + 3y = 6$

On subtracting, we get $x = 1$ and hence $y = 0$. Therefore, the point of intersection is (1, 0). The equation of the straight line cutting off equal intercepts is $\frac{x}{a} + \frac{y}{a} = 1$ (i.e.) $x + y = a$.

This straight line passes through (1, 0). Therefore, $1 + 0 = a$ (i.e.) $a = 1$. Hence, the equation of the required straight line is $x + y = 1$.

Example 2.20

Find the equation of the straight line concurrent with the lines $2x + 3y = 3$ and $x + 2y = 2$ and also concurrent with the lines $3x - y = 1$ and $x + 5y = 11$.

Solution

The point of intersection of the lines $2x + 3y = 3$ and $x + 2y = 2$ is obtained by solving the following two equations:

$$2x + 3y = 3 \tag{2.35}$$

$$x + 2y = 2 \tag{2.36}$$

(2.35) $\qquad 2x + 3y = 3$

(2.36) $\times 2$ $\qquad 2x + 4y = 4$

On subtracting, we get $y = 1$ and hence $x = 0$.

Therefore, the point of intersection is (0, 1).

$$3x - y = 1 \tag{2.37}$$

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$$x+5y=11$$

$$(2.37)\times 5 \qquad 15x-5y=5$$

$$(2.38)\times 1 \qquad x+5y=11$$

On adding, we get $16x = 16$ which gives $x = 1$ and hence $y = 2$. The point of intersection of the second pair of lines is $(1, 2)$. The equation of the line joining the two points $(0, 1)$ and $(1, 2)$ is

$$\frac{y-1}{x-0}=\frac{1-2}{0-1}=1 \Rightarrow y-1=x \quad \text{(i.e.)} \quad x-y+1=0.$$

Example 2.21

Find the angle between the lines $y=\sqrt{3}x+4$ and $y=\frac{1}{\sqrt{3}}x+2$.

Solution

The slope of the line $y=\sqrt{3}x+4$ is $\sqrt{3}$. Therefore, $m_1 = \tan\theta_1 = \sqrt{3}$ (i.e.) $\theta_1 = 60°$. The slope the line $y=\frac{1}{\sqrt{3}}x+2$ is $\frac{1}{\sqrt{3}}$. Therefore, $m_2 = \tan\theta_2 = \frac{1}{\sqrt{3}}$ (i.e.) $\theta_2 = 30°$. The angle between the lines is $\theta_1 - \theta_2 = 30°$.

Example 2.22

Find the equation of the perpendicular bisector of the line joining the points $(-2, 6)$ and $(4, -6)$.

Solution

The slope of the line joining the points $(-2, 6)$ and $(4, -6)$ is $m=\frac{6+6}{-2-4}=\frac{-12}{6}=-2$. Therefore, the slope of the perpendicular line is $\frac{1}{2}$.

The midpoint of the line joining the points $(-2, 6)$ and $(4, -6)$ is $\left(\frac{-2+4}{2},\frac{6-6}{2}\right)$ (i.e.) $(1, 0)$.

Therefore, the equation of the perpendicular bisector is $y-y_1=m(x-x_1)$ (i.e.) $y-0=\frac{1}{2}(x-1) \Rightarrow 2y=x-1$ or $x-2y-1=0$.

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Example 2.27

Find the equation of the line passing through the point (2, 3) and parallel to $3x - 4y + 5 = 0$.

Solution

The slope of the line $3x - 4y + 5 = 0$ is $\frac{3}{4}$.

Therefore, the slope of the parallel line is also $\frac{3}{4}$.

Hence the equation of the parallel line through (2, 3) is $y - y_1 = m(x - x_1)$

(i.e.) $y - 3 = \frac{3}{4}(x - 2) \Rightarrow 4y - 12 = 3x - 6.$ (i.e.) $3x - 4y + 6 = 0.$

Example 2.28

Find the equation of the line passing through the point (4, −5) and is perpendicular to the line $7x + 2y = 15$.

Solution

The slope of the line $7x + 2y = 15$ is $\frac{-7}{2}$.

Therefore, the slope of the perpendicular line is $\frac{+2}{7}$. The equation of the perpendicular line through (4, − 5) is $y - y_1 = m(x - x_1)$

$$\text{(i.e.)} \quad y + 5 = \frac{2}{7}(x - 4) \Rightarrow 7y + 35 = 2x - 8$$

$$\Rightarrow 2x - 7y = 43.$$

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