Introduction

A control system is a system, which provides the desired response by controlling the output. The followingfigureshows the simpleblock diagram ofacontrol system.

Here, the control system is represented by a single block. Since, the output is controlled by varying input, thecontrol system got this name. We will vary this input with some mechanism. In the next section on open loopand closed loop control systems, we will study in detail about the blocks inside the control system and how tovarythis input in order to get thedesired response.

Examples −Trafficlightscontrolsystem,washingmachine

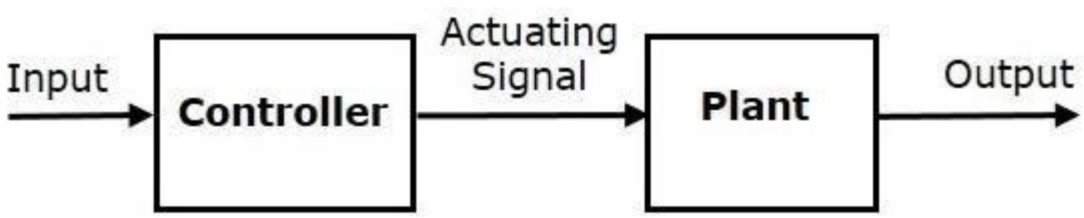
Traffic lights control system is an example of control system. Here, a sequence of input signal is applied to thiscontrol system and the output is one of the three lights that will be on for some duration of time. During thistime, the other two lights will be off. Based on the traffic study at a particular junction, the on and off times ofthe lights can be determined. Accordingly, the input signal controls the output. So, the traffic lights controlsystemoperates on timebasis.

OpenLoop andClosed LoopControl Systems

ControlSystemscanbeclassifiedasopenloopcontrolsystemsandclosedloopcontrolsystemsbasedonthefeedback path.

In open loop control systems, output is not fed-back to the input. So, the control action is independent of thedesiredoutput.

Thefollowing figureshows theblock diagramof theopen loop control system.



Here, an input is applied to a controller and it produces an actuating signal or controlling signal. This signal isgiven as an input to a plant or process which is to be controlled. So, the plant produces an output, which iscontrolled. The traffic lights control system which we discussed earlier is an example of an open loop controlsystem.

In closed loop control systems, output is fed back to the input. So, the control action is dependent on the desiredoutput.

TransferFunction

Transferfunctionofalineartime-invariantsystemisdefinedastheratiooftheLaplacetransform of output variable to the Laplace transform of input variable assuming all the initialconditions to be zero. The figure 1a shows the system in time domain whereas figure 1b showsthesysteminLaplacedomain.

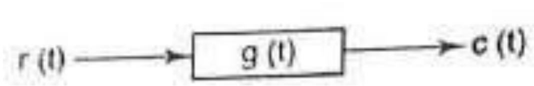


Figure1a. systemintimedomain

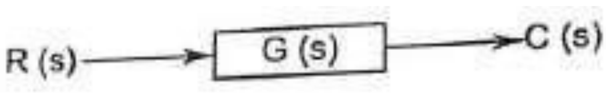


Fig1b.systeminLaplacedomain.

Figure1.Transfer Functionofasystem

IfG(s)bethetransferfunctionofthesystem, wecanwritemathematicallyas

$$G(s) = \frac{\text{Laplacetransformof}}{\text{outputLaplacetransformofi nput}} \text{ (allinitialconditionsarezero)}$$

$$= \frac{C(s)}{R(s)} \text{ (allinitialconditionsarezero)} \ldots\ldots (1)$$

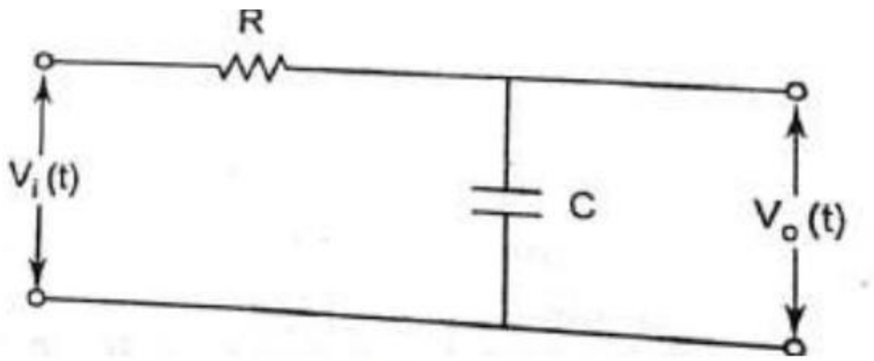


Figure 2

Solution- Let i(t) be the current flowing through the circuit using KVL we can write

$$V_i(t) = Ri(t) + \frac{1}{C}\int_{-\infty}^{t} i(t)dt$$

$$\text{And } V_o(t) = \frac{1}{C}\int_{-\infty}^{t} i(t)dt$$

Taking Laplace transfer of the above equation by assuming zero initial condition, we get

$$V_i(s) = RI(s) + \frac{1}{sC} I(s)$$

$$\text{And } V_o(s) = \frac{1}{sC} I(s)$$

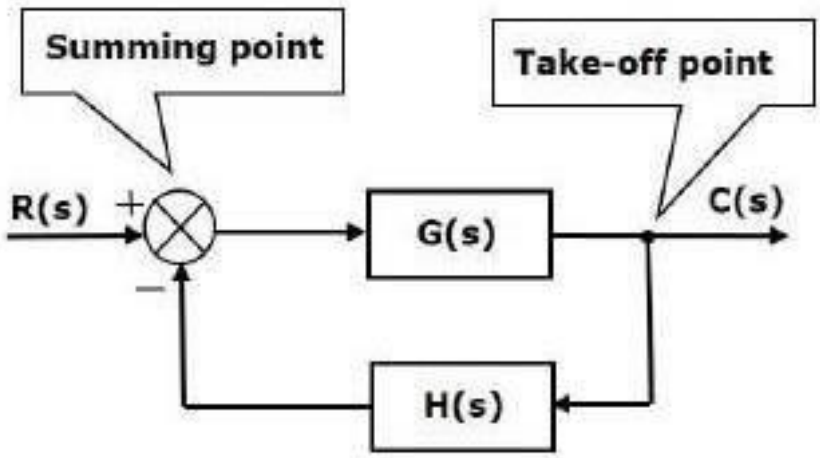
$$\therefore Transfer\ Function = G(s) = \frac{Vo(s)}{Vi(s)} = \frac{1}{1+sCR}$$

BlockDiagrams

Block diagram is the pictorial representation ofsystem.It consists of a single block or acombinationofblocks.Eachblockis afunctionalblock.

BasicElementsofBlock Diagram

Thebasicelementsofablockdiagramareablock,thesummingpointandthetake-offpoint.Let us consider the block diagram of a closed loop control system as shown in the followingfiguretoidentifytheseelements.



The above block diagram consists of two blocks having transfer functions G(s) and H(s). It isalsohavingonesummingpointandonetake-offpoint.Arrowsindicatethedirectionofthe flow of signals. Let us now discuss these elements one by

one.Block

Thetransferfunctionofacomponentisrepresentedbyablock.Blockhassingleinputand singleoutput.

ThefollowingfigureshowsablockhavinginputX(s),outputY(s)andthetransferfunctionG(s).

,

TransferFunction,G(s)=Y(s)/X(s)

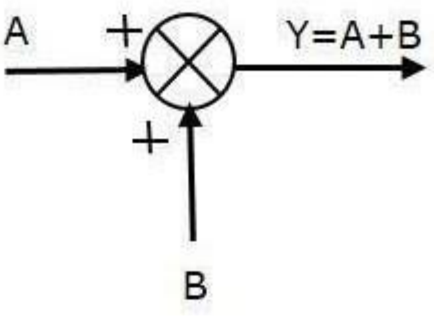
⇒Y(s)=G(s)X(s)

Outputoftheblockisobtainedby multiplyingtransferfunctionoftheblockwithinput.Summing Point

Thesummingpointisrepresentedwithacirclehavingcross(X)insideit.Ithastwoormore inputsandsingleoutput.Itproducesthealgebraicsumoftheinputs.Italsoperformsthe summationorsubtractionorcombinationofsummationandsubtractionoftheinputsbasedonthepolarit yoftheinputs.Letus seethesethreeoperationsonebyone.

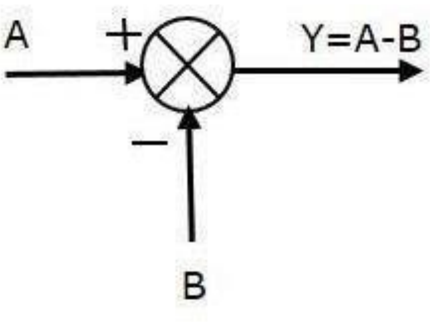
The following figure shows the summing pointwith two inputs (A, B) and one output (Y).Here, theinputs A andB have a positive sign. So, the summing pointproduces the output, Yassum ofAandB.

i.e.=A+B.



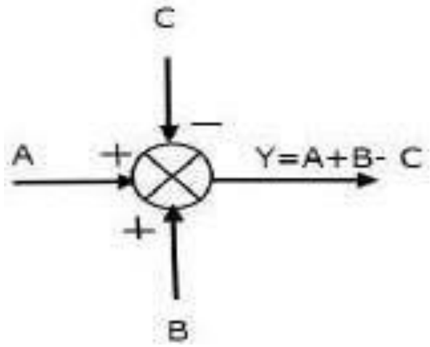
The following figure shows the summing pointwith two inputs (A, B) and one output (Y).Here, the inputs A and B are having opposite signs, i.e., A is having positive sign and B ishaving negative sign. So, the summing point produces the outputY as the difference of A andB.

Y=A+(-B)=A-B.



The following figure shows the summing point with three inputs (A, B, C) and one output (Y).Here, the inputs A and B are having positive signs and C is having a negative sign. So, thesummingpointproduces theoutputYas

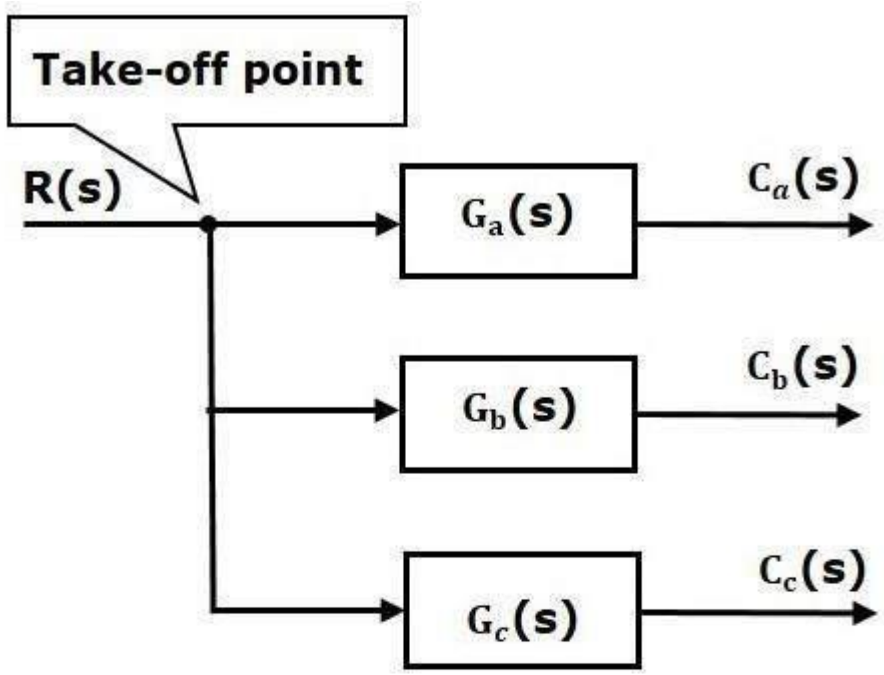
$Y=A+B+(-C)=A+B-C$.



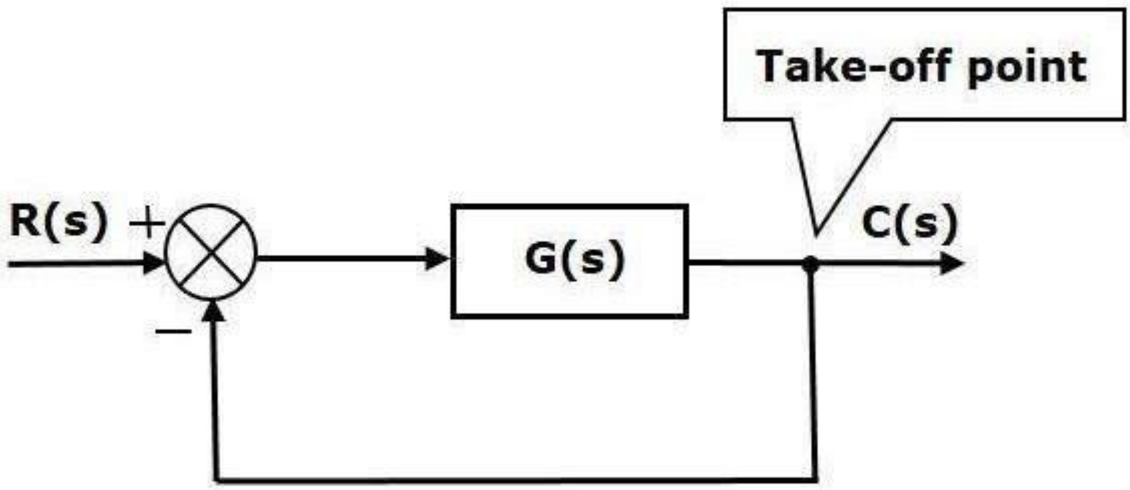
Take-offPoint

The take-off point is a point from which the same input signal can be passed through more thanone branch. That means with the help of take-off point, we can apply the same input to one ormoreblocks,summingpoints.

In the following figure, the take-off point is used to connect the same input, R(s) to two moreblocks.



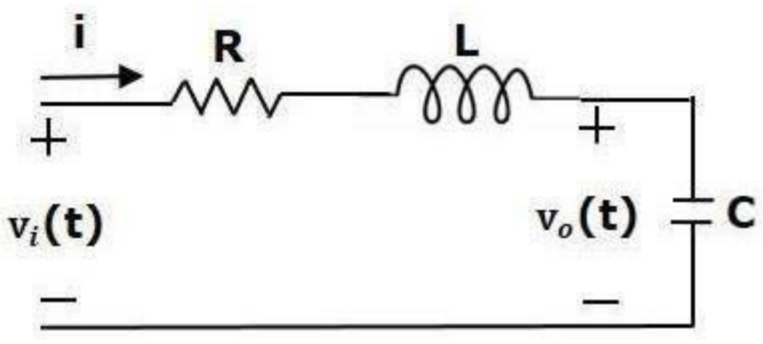
In the following figure, the take-off point is used to connect the output C(s), as one of the inputstothesummingpoint.



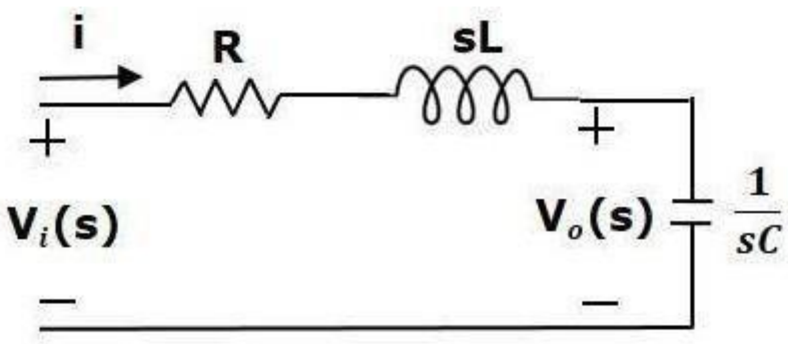
BlockDiagramRepresentationofElectricalSystems

In this section, let us represent an electrical system with a block diagram. Electrical systemscontainmainlythreebasic elements—resistor,inductorandcapacitor.

Consider a series of RLC circuit as shown in thefollowingfigure. Where, $V_i(t)$ and $V_o(t)$ aretheinputandoutputvoltages.Leti(t)bethecurrentpassingthroughthecircuit.Thiscircuitisintimedomain.



By applying the Laplace transform to this circuit, will get the circuit in s-domain. The circuit isasshowninthefollowingfigure.



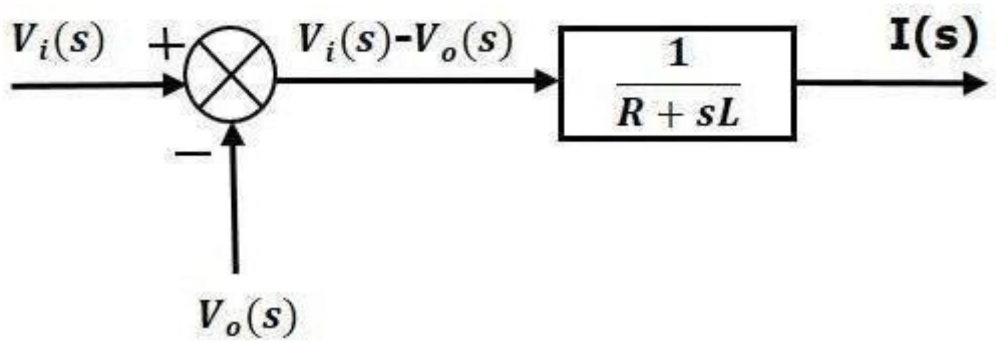
Fromtheabovecircuit,wecanwrite

$$I(s)=[Vi(s)-Vo(s)]/R+sL$$

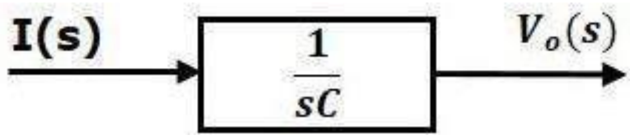
$\Rightarrow I(s)=\{1/R+sL\}\{Vi(s)-Vo(s)\}$ (Equation1)
$Vo(s)=(1/sC)I(s)$ (Equation2)

Let us now draw the block diagrams for these two equations individually. And then combinethose block diagrams properly in order to get the overall block diagram of series of RLC Circuit(s-domain).

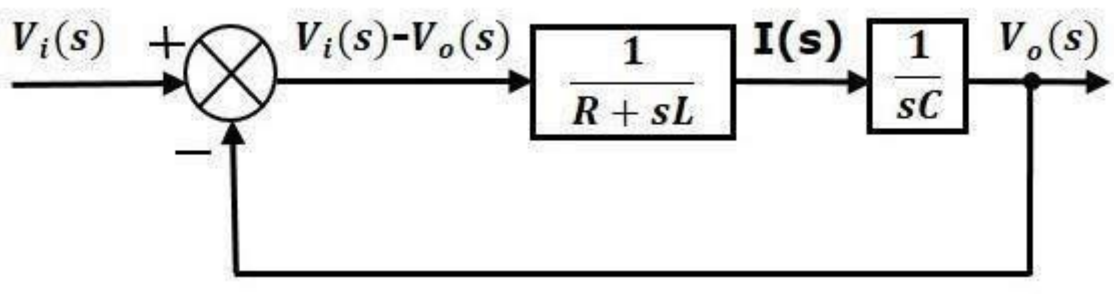
Equation 1 can be implemented with a block having the transfer function, 1/R+sL.The inputandoutputofthis blockare {Vi(s)−Vo(s)} and I(s). Werequireasummingpoint to 1isshowninthefollowingfigure.



Equation 2 can be implemented with a block having transfer function, 1/sC. The input andoutput of this block are I(s) and Vo(s). The block diagram of Equation 2 is shown in thefollowingfigure.



The overall block diagram of the series of RLC Circuit (s-domain) is shown in the followingfigure.



Similarly,youcandrawthe blockdiagramofanyelectricalcircuitorsystem justbyfollowingthissimpleprocedure.

- Convertthetime domainelectricalcircuitintoans-domainelectricalcircuitbyapplyingLaplacetransform.

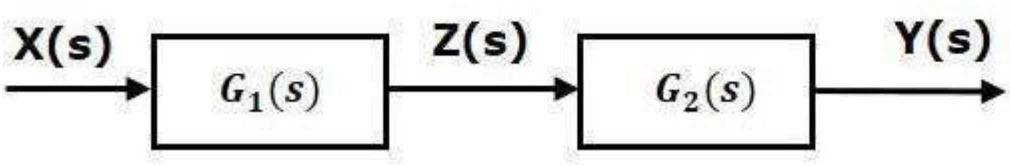
- Writedowntheequationsforthecurrentpassingthroughallseriesbranchelementsandvoltagea crossall shuntbranches.
- Drawtheblockdiagramsforalltheaboveequationsindividually.
- Combinealltheseblockdiagramsproperlyinordertogettheoverallblockdiagramoftheelectric alcircuit(s-domain).

Blockdiagramreductionrules:

Therearethreebasictypesofconnections

betweentwoblocks.Rule1:Series Connection

Series connection is also called cascade connection. In the following figure, two blocks havingtransferfunctionsG1(s)andG2(s)are connectedinseries.



Forthiscombination,wewillgettheoutputY(s)as

$$Y(s)=G2(s)Z(s)$$

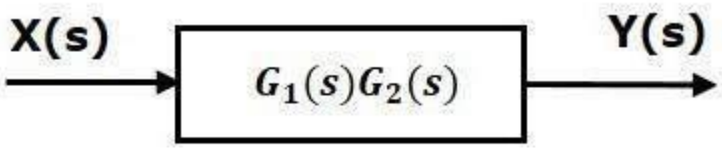
Where,Z(s)=G1(s)X(s)

$$\Rightarrow Y(s)=G2(s)[G1(s)X(s)]=G1(s)G2(s)X(s)$$

$$\Rightarrow Y(s)=\{G1(s)G2(s)\}\ X(s)$$

Compare this equation with the standard form of the outputequation, Y(s)=G(s)X(s).Where,G(s)=G1(s)G2(s).

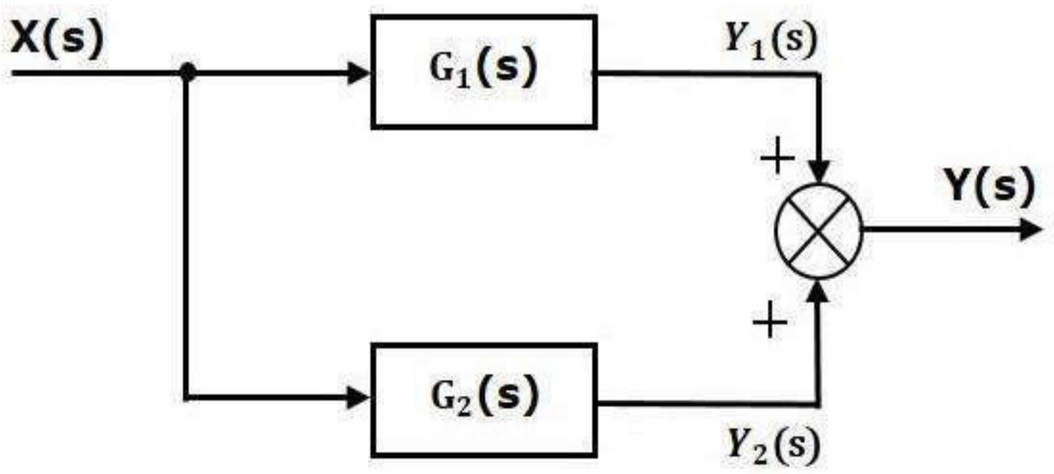
That means we can represent the series connection of two blocks with a single block. Thetransfer function of this single block is the product of the transfer functions of those two blocks.Theequivalentblockdiagramisshownbelow.



Similarly, you can represent series connection of ‘n’ blocks with a single block. The transferfunctionofthissingleblockisthe productofthetransferfunctionsofallthose‘n’ blocks.

Rule2:ParallelConnection

The blocks which are connected inparallel will have the same input. In the following figure,two blocks having transfer functions G1(s) and G2(s) are connected in parallel. The outputs ofthesetwoblocksareconnectedto thesummingpoint.



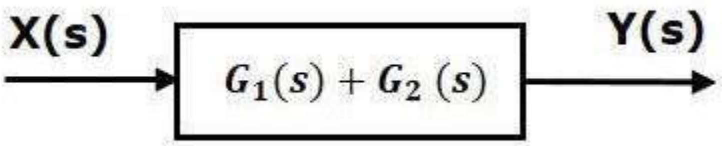
Forthiscombination,wewillgettheoutputY(s)as

$$Y(s) = Y1(s)+Y2(s)$$

Where, Y1(s)=G1(s)X(s))and Y2(s) =G2(s)X(s)

$$\Rightarrow Y(s)=G1(s)X(s)+G2(s)X(s)=\{G1(s)+G2(s)\}\ X(s)$$

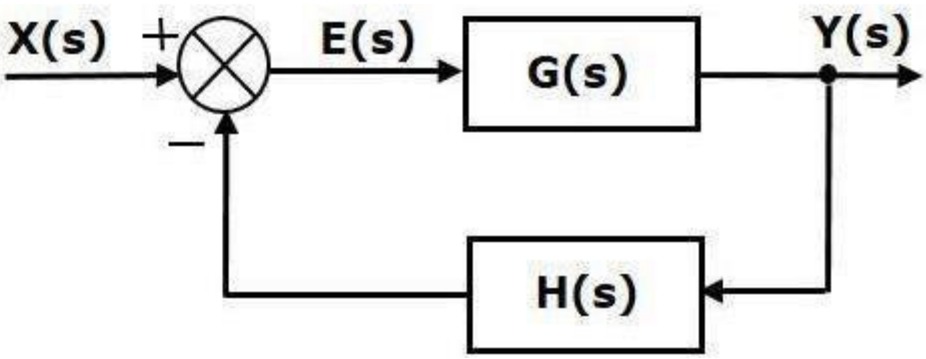
Compare this equation with the standard form of the output equation,Y(s) =G(s)X(s)
Where,G(s) =G1(s)+G2(s)
That means we can represent the parallel connection of two blocks with a single block. Thetransferfunctionofthissingleblockisthesumofthetransferfunctionsofthosetwoblocks.Theequivalentblockdiagramis shown below.



Similarly, you can represent parallel connection of 'n' blocks with a single block. The transferfunction of this single block is the algebraic sum of the transfer functions of all those 'n'blocks.

Rule3:FeedbackConnection

As we discussed in previous chapters, there are two types of feedback — positive feedback andnegative feedback. The following figure shows negative feedback control system. Here, twoblockshaving transferfunctionsG(s)and H(s)formaclosedloop.



Theoutputofthesumming pointis-

$$E(s)=X(s) - H(s)Y(s)$$

Theoutput Y(s)is-

$$Y(s)=E(s)G(s)$$

SubstituteE(s)valueintheaboveequation.

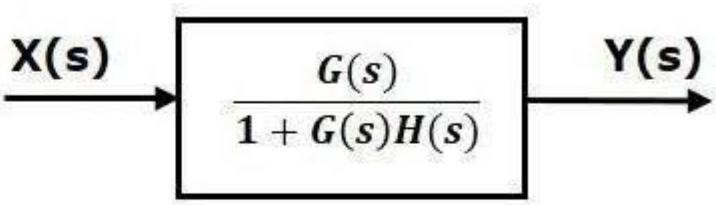
$$Y(s)=\{X(s)-H(s)Y(s)\}G(s)\}$$

$$Y(s)\{1+G(s)H(s)\}=X(s)G(s)$$

$$\Rightarrow Y(s)/X(s)=G(s)/[1+ G(s)H(s)]$$

Therefore,thenegativefeedbackclosedlooptransfer functionisG(s)/[1+G(s)H(s)]

This means we can represent the negative feedback connection of two blocks with a singleblock. The transfer function of this single block is the closed loop transfer function of thenegativefeedback.The equivalentblockdiagramis shownbelow.



Similarly, you can represent the positive feedback connection of two blocks with a single block.The transfer function of this single block is the closed loop transfer function of the positivefeedback,i.e.,G(s)/[1−G(s) H(s)]

Rule4:BlockDiagramAlgebraforSummingPoints

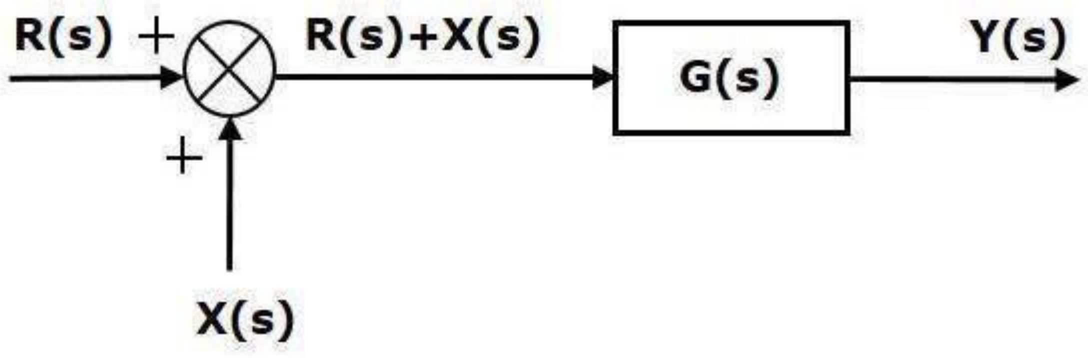
Therearetwopossibilitiesofshiftingsumming pointswithrespect toblocks−

- Shiftingsummingpoint aftertheblock
- Shiftingsummingpoint beforetheblock

Letusnowseewhatkindofarrangementsneedtobedone intheabovetwocasesonebyone.

Rule4a:ShiftingSummingPointaftertheBlock

Considertheblockdiagramshowninthefollowingfigure.Here,thesummingpointispresentbeforetheb lock.

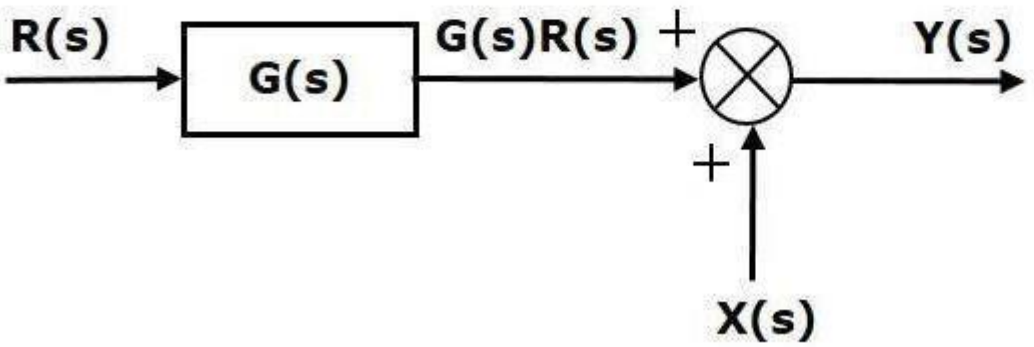


Summing point has two inputs R(s) and X(s). The output of it is {R(s)+X(s)}So,theinputtothe block G(s)is{R(s)+X(s)}andtheoutput ofitis –

$$Y(s)=G(s)\{R(s)+X(s)\}$$

$$\Rightarrow Y(s)=G(s)R(s)+G(s)X(s) \text{(Equation 1)}$$

Now,shiftthesummingpointaftertheblock.Thisblockdiagramisshowninthefollowingfigure.



Output of the block G(s) is

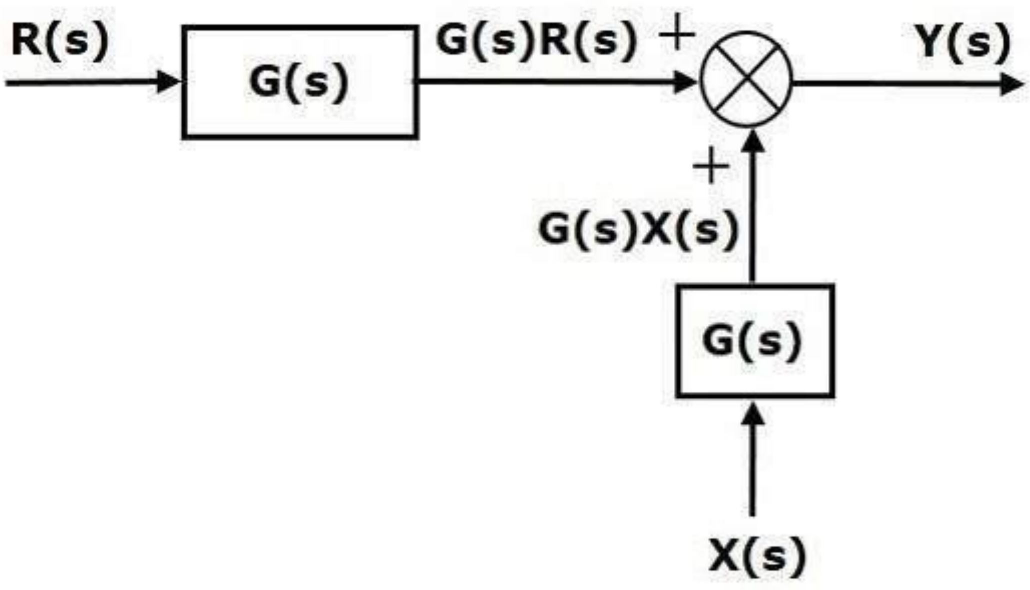
G(s)R(s)Theoutputofthesummingpo

intis

$Y(s)=G(s)R(s)+X(s)$ (Equation2)

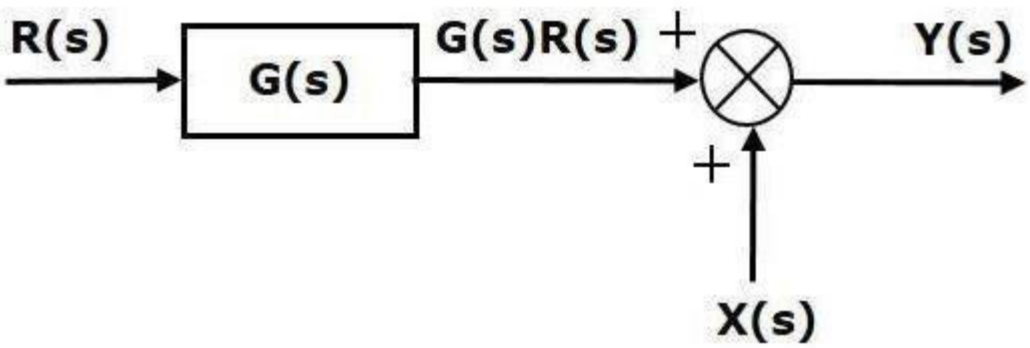
CompareEquation1andEquation2.

The first term ‘G(s)R(s)′ is same in both the equations. But, there is difference in the secondterm. In order to get the second term also same,we require one more block G(s). Itis havingthe input X(s) and the output of this block is given as input to summing point instead of X(s).Thisblockdiagramisshown inthefollowingfigure.



Rule4b:ShiftingSumming PointBeforetheBlock

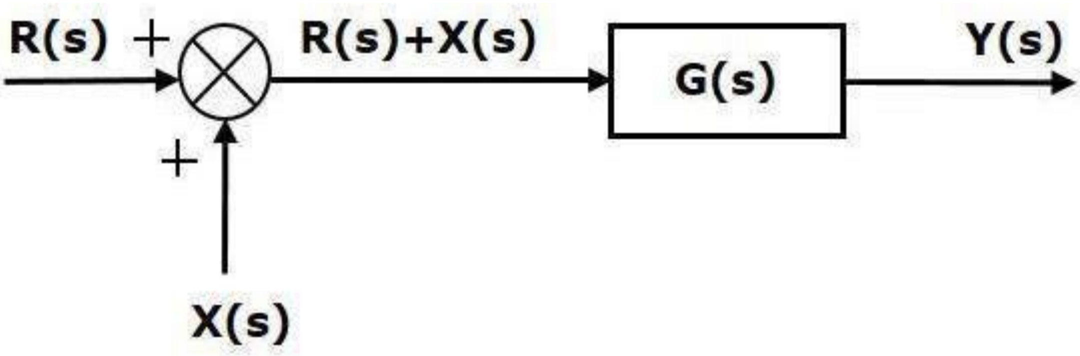
Considertheblockdiagramshowninthefollowingfigure.Here,thesummingpointispresentaftertheblock.



Outputofthisblock diagramis-

$Y(s) = G(s)R(s)+X(s)$ (Equation3)

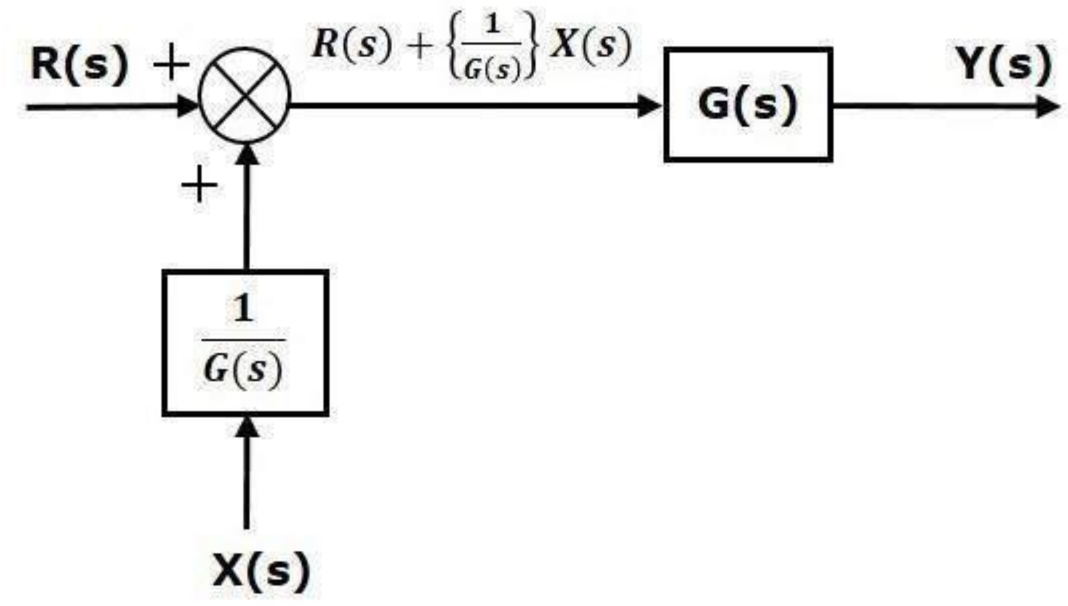
Now,shiftthesummingpointbeforetheblock.Thisblockdiagramisshowninthefollowingfigure.



Outputofthisblock diagramis-

Y(S)=G(s) R(s)+G(s)X(s) (Equation 4)CompareEquation3 andEquation4,

The first term ‘G(s) R(s)′ is same in both equations. But, there is difference in the second term.In order to get the second term also same, we require one more block 1/G(s). It is having theinput X(s) and the output of this block is given as input to summing point instead of X(s). Thisblockdiagramis shown inthefollowingfigure.



Rule5:BlockDiagramAlgebraforTake-offPoints

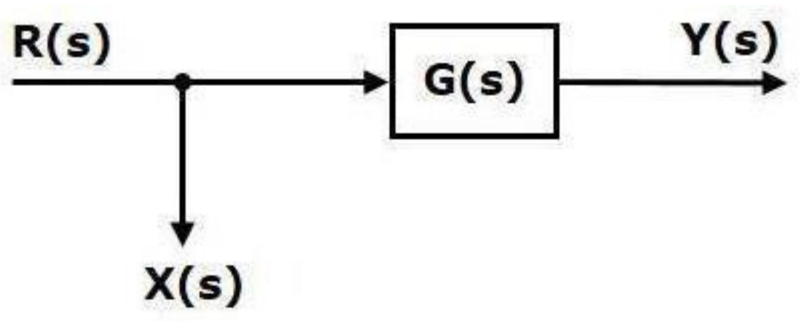
Therearetwopossibilitiesofshiftingthetake-offpointswithrespecttoblocks−

- Shiftingtake-offpointafter theblock
- Shiftingtake-offpointbeforetheblock

Letusnowseewhatkindofarrangementsaretobedoneintheabovetwocases,onebyone.Rule5a:Shiftin

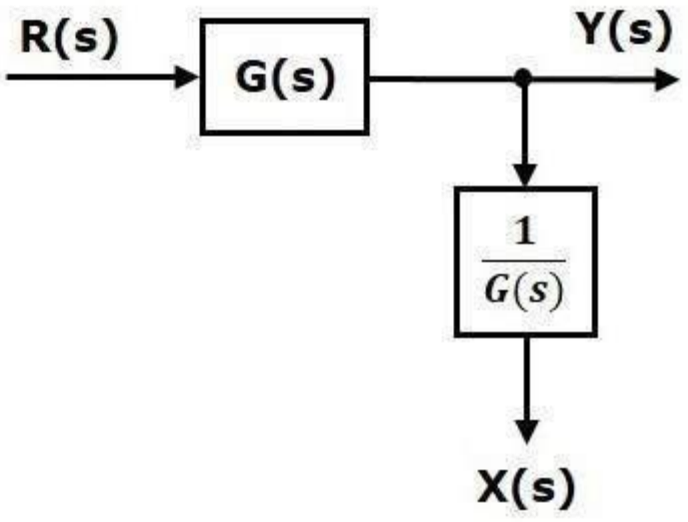
gTake-offPointaftertheBlock

Considertheblockdiagramshowninthefollowingfigure.Inthiscase,thetake-offpointis presentbeforetheblock.



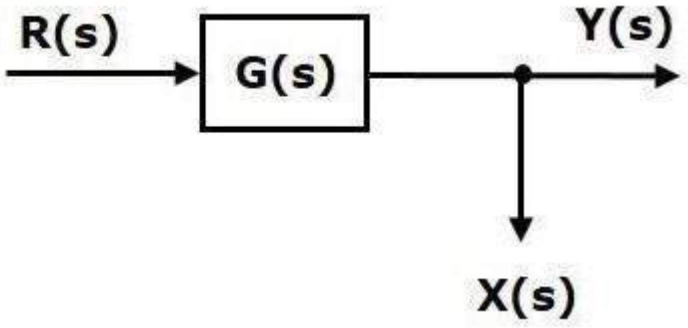
Here,X(s)=R(s) andY(s) =G(s)R(s)
Whenyoushiftthetake-offpointaftertheblock,theoutputY(s)willbesame.But,thereisdifferenceinX(s)value.So,inordertoget thesameX(s)value,werequireonemore

block 1/G(s). It is having the input Y(s) and the output is X(s). This block diagram is shown inthefollowingfigure.



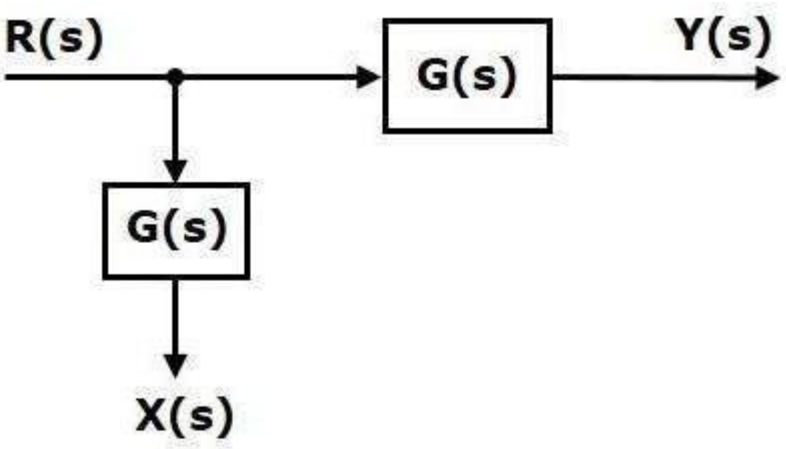
Rule5b:ShiftingTake-offPointBeforetheBlock

Consider the block diagram shown in the following figure. Here, the take-off point is presentaftertheblock.



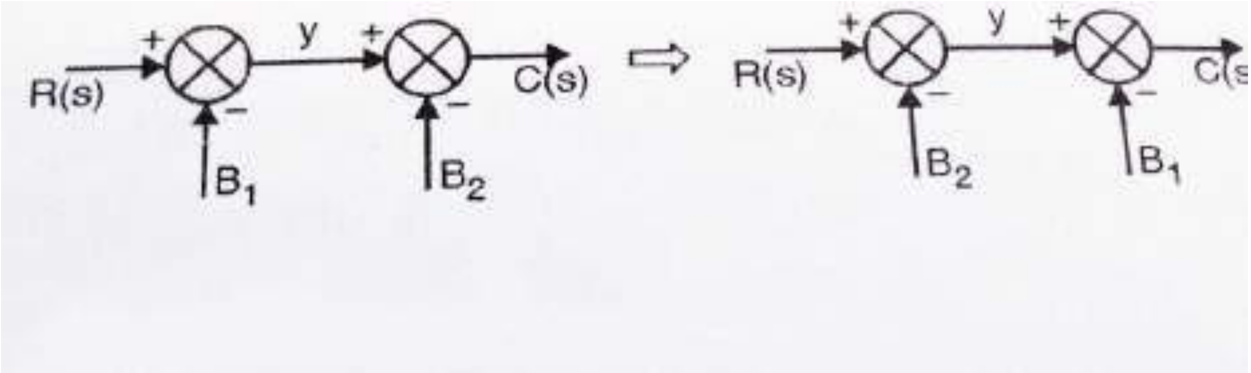
Here, X(s)=Y(s) =G(s)R(s)

When you shift the take-off point before the block, the output Y(s) will be same. But, there isdifference in X(s) value. So, in order to get same X(s) value, we require one more block G(s). Itis having the input R(s) and the output is X(s). This block diagram is shown in the followingfigure.



Rule6:AssociativeLaw ForSummingPoint

Thiscanbe betterexplainedbytaking belowdiagram



Y=R(s)– B1
C(s)=y–B2 =R(s)–B1 –B2
Thislawis applicableonlytosummingpointswhichareconnecteddirectlytoeachother.Note: If there is a block present between two summing points(and hence they are not connecteddirectly)thenthis rulecan'tbeapplied.

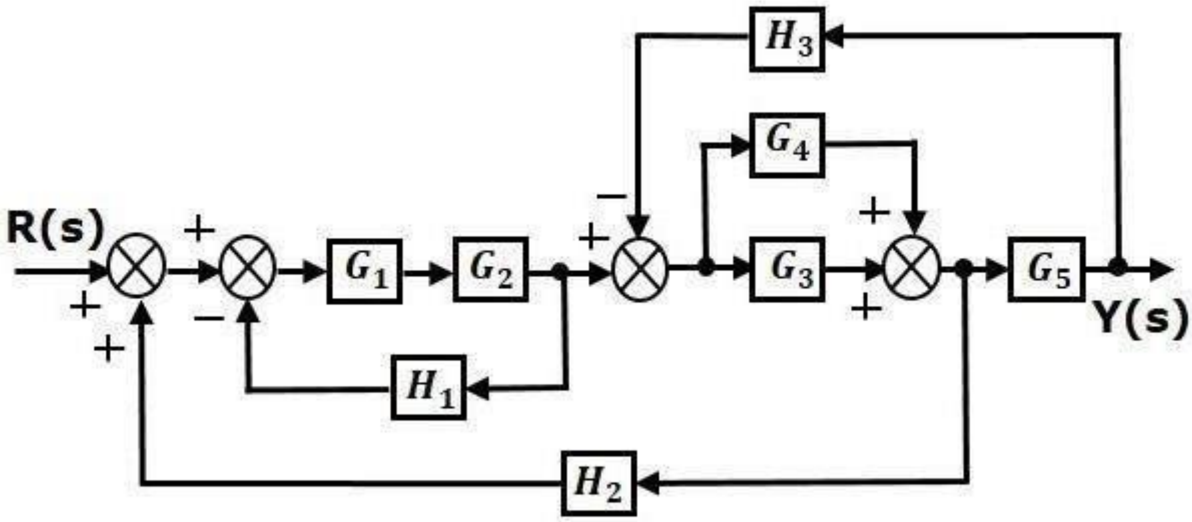
Procedureforfinding TFbyusingBlockDiagramReductionRules

Followtheserulesforsimplifying(reducing)theblockdiagram,whichishavingmany blocks,summingpoints andtake-offpoints.

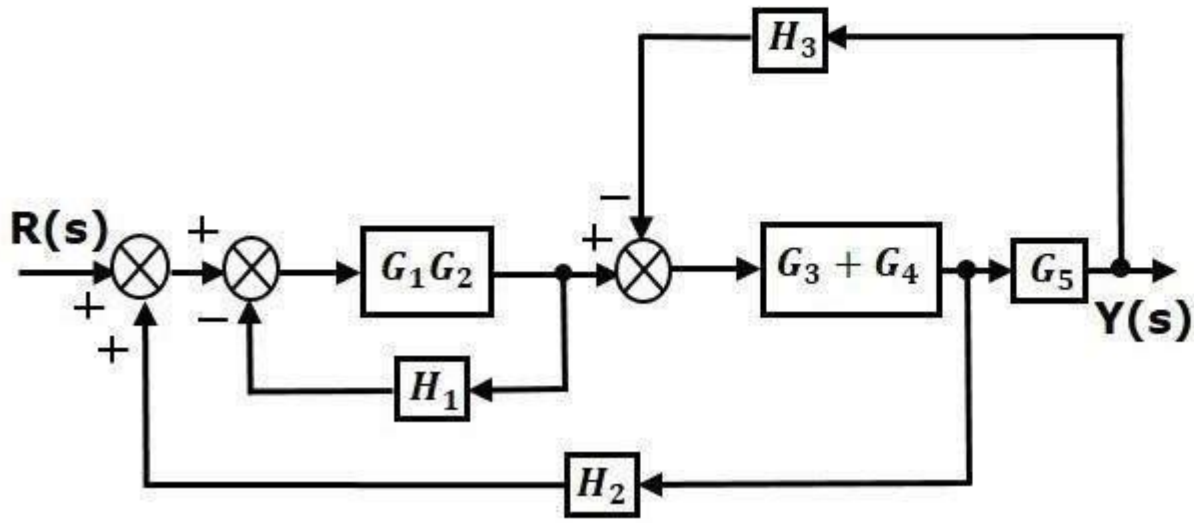
- Rule1−Checkfortheblocksconnected inseriesandsimplify.
- Rule2 −Checkfortheblocksconnected inparalleland simplify.
- Rule3−Checkfortheblocksconnectedinfeedbackloopandsimplify.
- Rule4−Ifthereisdifficultywith take-offpointwhilesimplifying,shiftittowardsrightorleftofthegivenblockwhichoneis suitable.
- Rule5 −Ifthereisdifficultywithsummingpointwhilesimplifying,shiftittowardsrightorleftofthegivenblockwhichoneis suitable.
- Rule6−Repeattheabovestepstillyougetthesimplifiedform,i.e.,singleblock.

Example

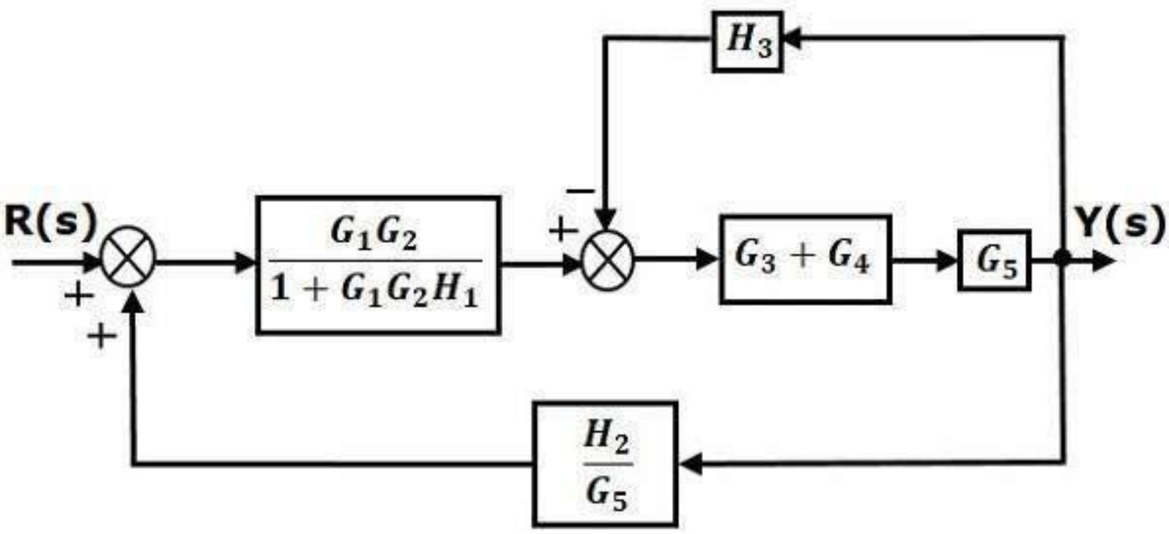
Considertheblockdiagramshowninthefollowingfigure.Letussimplify(reduce)thisblockdiagramusingtheblockdiagramreductionrules.



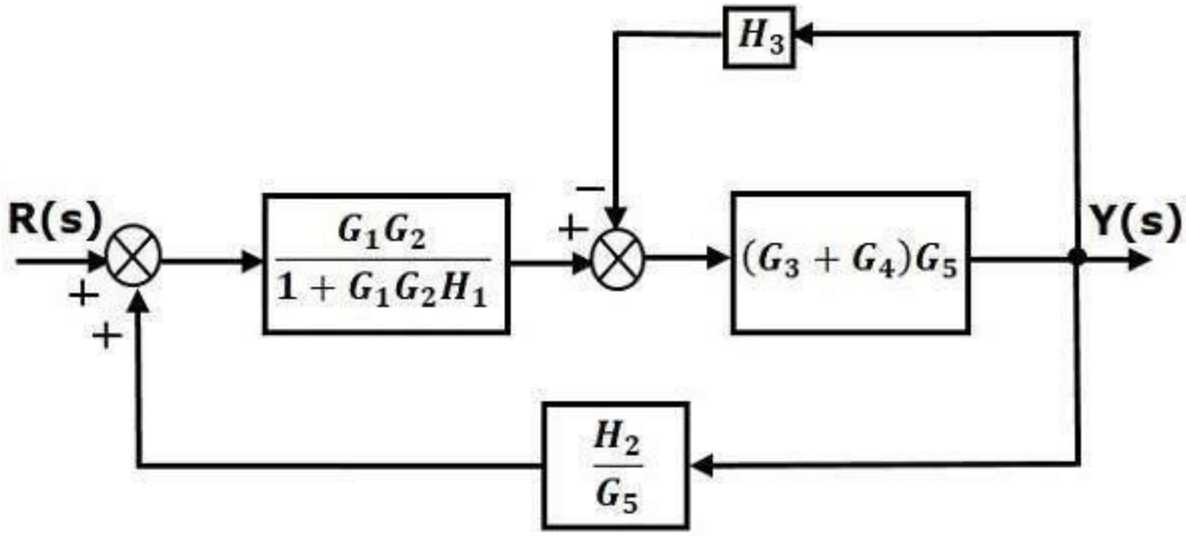
Step1 −UseRule1forblocks G1andG2.UseRule2forblocksG3 and G4.Themodifiedblockdiagramis shown inthefollowingfigure.



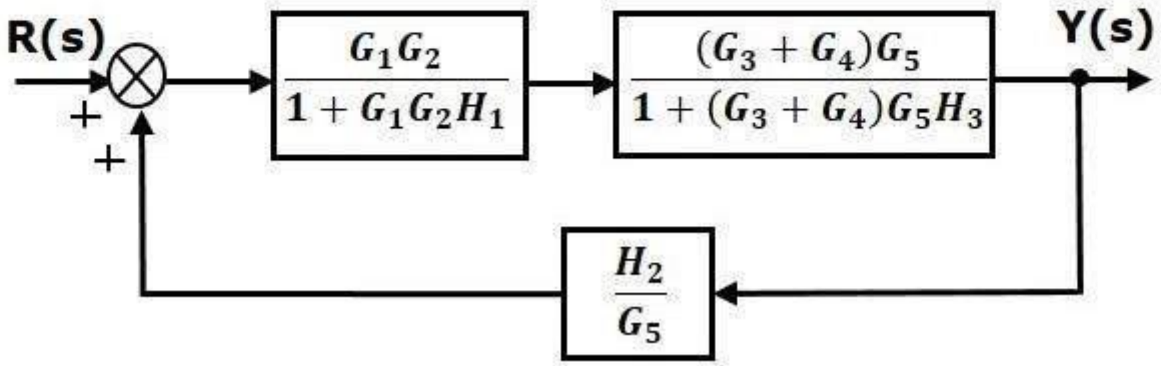
Step2−UseRule3forblocksG1G2and H1.UseRule4forshiftingtake-offpointaftertheblockG5.Themodifiedblockdiagramisshowninthefollowingfigure.



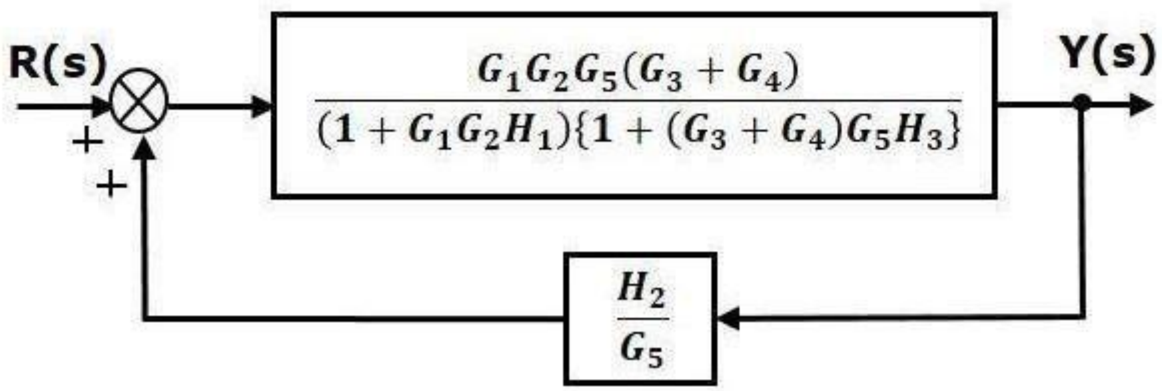
Step3−UseRule1forblocks(G3+G4) and G5.Themodifiedblockdiagramisshowninthefollowingfigure.



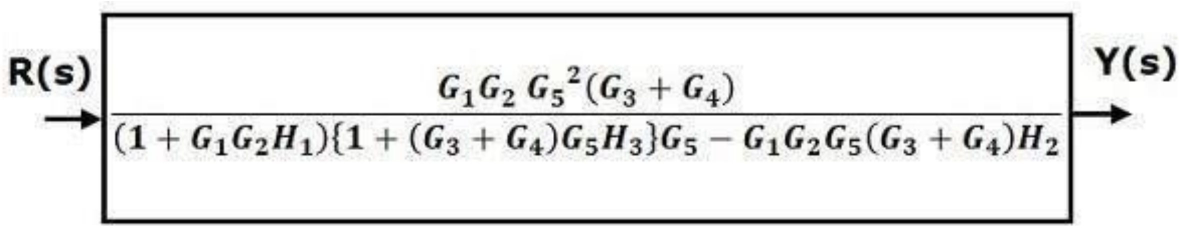
Step4−UseRule3forblocks (G3+G4)G5andH3.Themodifiedblockdiagramisshowninthefollowingfigure.



Step5−UseRule1forblocksconnectedinseries.Themodifiedblockdiagramisshowninthefollowingfigure.



Step6−UseRule3forblocksconnectedinfeedbackloop.Themodifiedblockdiagramisshowninthefollowingfigure.Thisis the simplified blockdiagram.



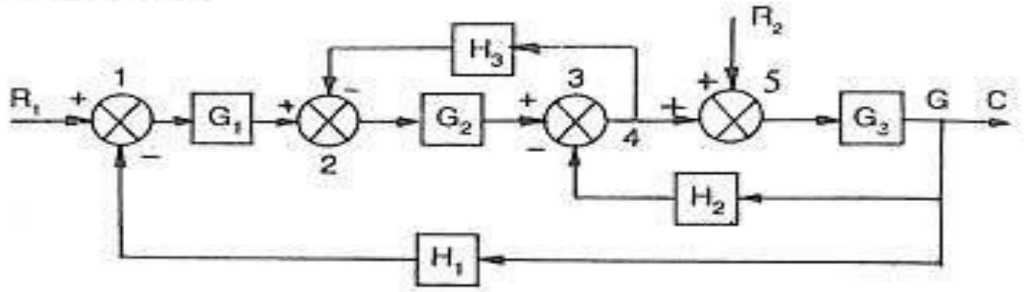
Therefore,thetransferfunctionofthesystemis

$Y(s)/R(s)=G_1G_2G_5{}^2(G_3+G_4)/(1+G_1G_2H_1)\{1+(G_3+G_4)G_5H_3\}G_5-G_1G_2G_5(G_3+G_4)H_2$

Note − Follow these steps in order to calculate the transfer function of the blockdiagramhavingmultipleinputs.

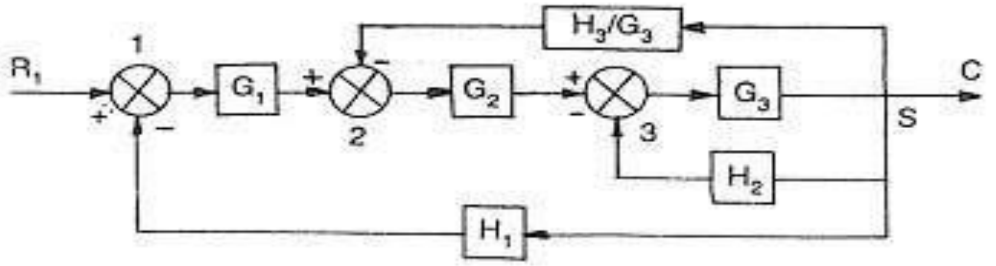
- Step1−Findthetransferfunctionofblockdiagrambyconsideringoneinputatatimeandmaketh e remaininginputsaszero.
- Step2−Repeatstep1forremaininginputs.
- Step3−Gettheoveralltransfer functionbyaddingallthosetransferfunctions.

Problem Evaluate $\frac{C}{R_1}$ and $\frac{C}{R_2}$ for a system whose block diagram representation is shown in Fig. R_1 is the input to summing point No. 1.

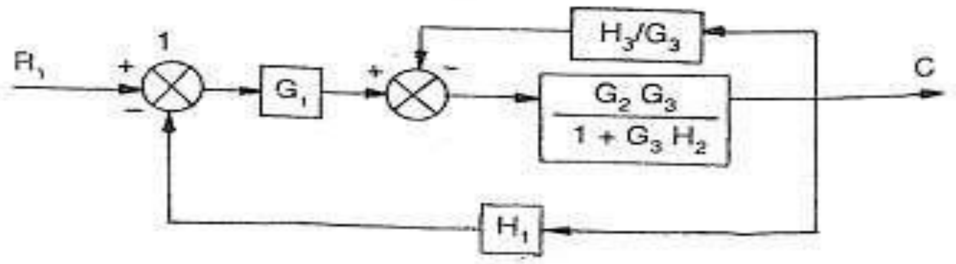


Solution

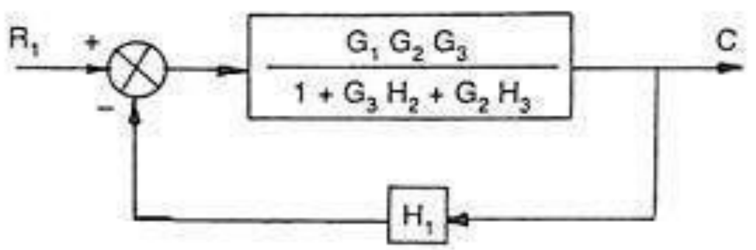
Evaluation of C/R₁ Assume $R_2 = 0$. Therefore summing point No. 5 can be removed. Shift take off point No. 4 beyond block G_3



Eliminate the feedback loop between points **3 and 6**



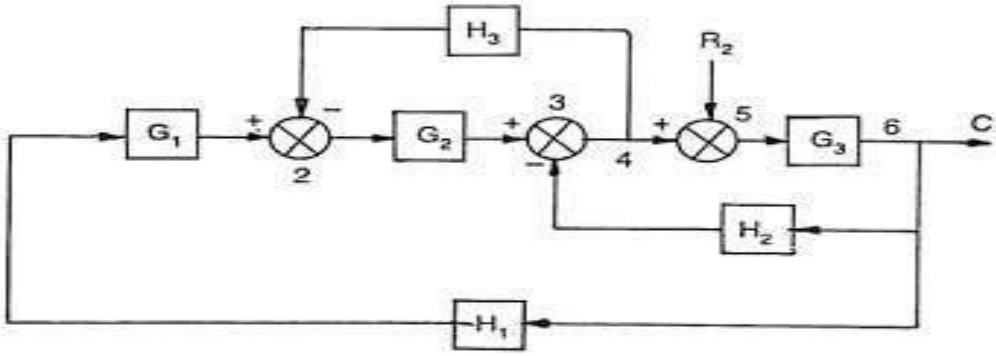
Eliminating the feed back loop again



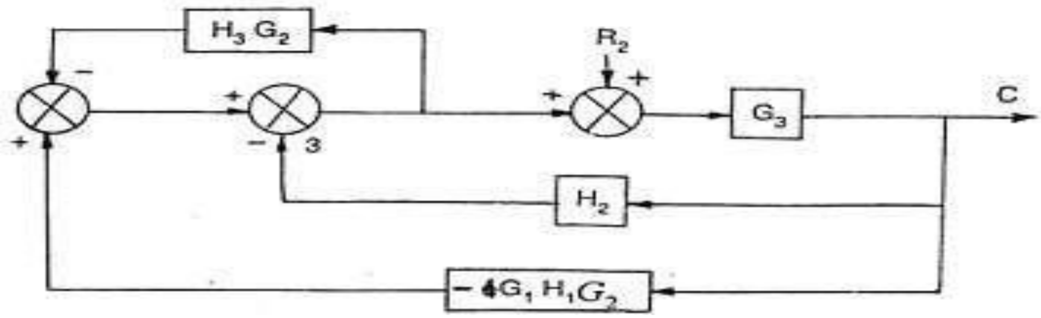
$$\frac{C}{R_1} = \frac{G_1G_2G_3}{1 + G_3H_2 + H_3G_2 + G_1G_2G_3H_1} \quad \textbf{Ans.}$$

Evaluation of C/R_2

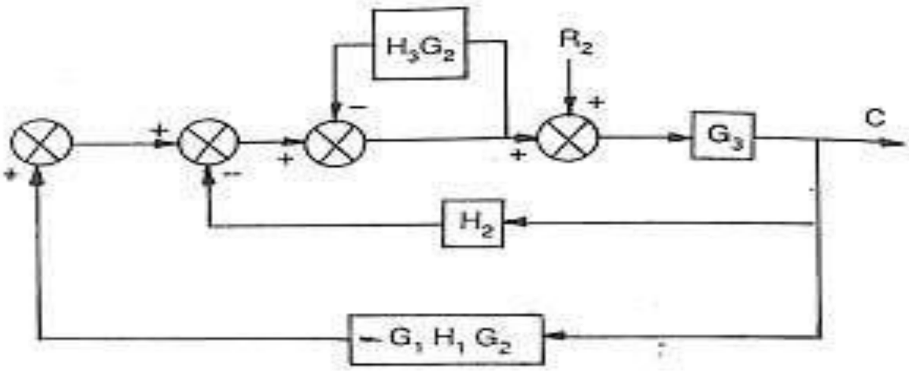
Assume $R_1 = 0$. Thus summing point No. 1 can be removed



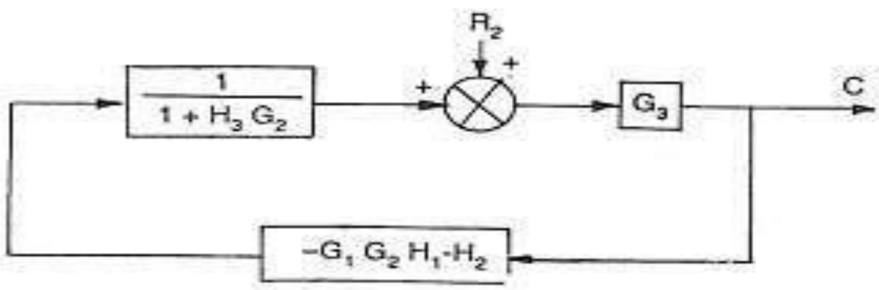
Shifting the summing point No. 2 and rearranging beyond G_2



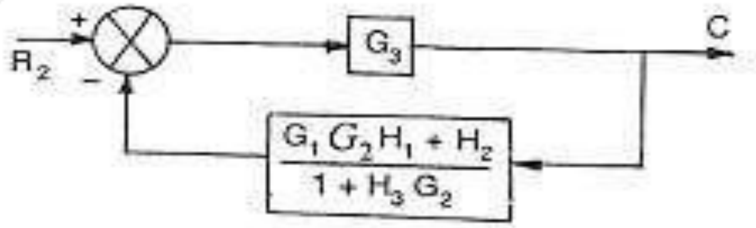
Rearranging, we get



Rearranging and eliminating the feedback loop



Rearranging,



Eliminating the feedback loop, we get

$$\frac{C}{R_2} = \frac{G_3(1 + H_3G_2)}{1 + H_3G_2 + G_3(G_1G_2H_1 + H_2)} \qquad \text{Ans.}$$

The block diagram reduction process takes more time for complicated systems. Because, wehave to draw the (partially simplified) block diagram after each step. So, to overcome thisdrawback,usesignalflowgraphs (representation).

SignalFlowGraphs

Signal flow graph is a graphical representation of algebraic equations. In this chapter, let usdiscuss the basic concepts related signal flow graph and also learn how to draw signal flowgraphs.

Characteristics of SFG: SFG is a graphical representation of the relationship between thevariablesofasetoflinearalgebraicequations.Itdoesn'trequireanyreductiontechniqueorprocess.

- It representsanetworkinwhichnodesareusedfortherepresentationofsystemvariablewhichis connectedbydirectbranches.
- SFG isadiagramwhichrepresentsasetofequations.Itconsistsofnodesandbranchessuchthateach branchofSFGhavinganarrowwhichrepresents theflowofthesignal.
- Itisonlyapplicabletothelinearsystem.

TerminologyusedinSFG

Nodesandbranchesarethebasicelementsofsignalflowgraph.

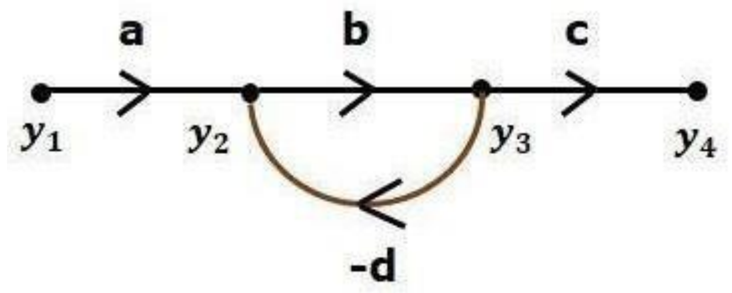
1. Node

Nodeisapointwhich representseitheravariableor asignal.Therearethreetypesof nodes— inputnode,outputnodeandmixednode.

- InputNodeorsource−Itisanode,whichhasonlyoutgoingbranches.
- OutputNodeorsink−Itisanode, whichhasonlyincomingbranches.
- MixedNode−Itisanode, whichhasbothincomingandoutgoing branches.

Example

Letusconsiderthefollowing signalflowgraphtoidentifythesenodes.



- Thenodespresentinthissignal flowgraphare y_1,y_2, $y_3$andy_4.
- $y_1$and$y_4$aretheinputnodeandoutputnoderespectively.
- $y_2$and $y_3$aremixed nodes.

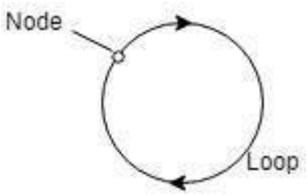
2. Branch

Branch is a line segment which joins two nodes. It has both gain and direction. For example,therearefourbranchesintheabovesignalflowgraph.Thesebrancheshavegainsofa,b,cand-d.

3. ForwardPath

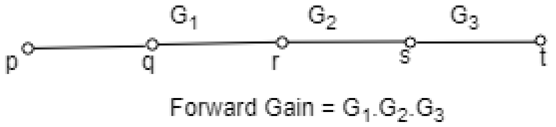
Itisapathfromaninputnodetoanoutputnodeinthedirectionofbrancharrow.

4. Loop:Itisapaththatstartsandendsatthesamenode.

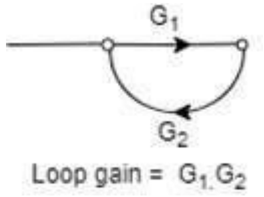


5. Non-touchingloop: Loopissaidtobenon-touchingiftheydonothaveanycommon node.

6. Forwardpathgain:Aproductofall branchesgainalongtheforwardpathiscalledForwardpathgain.



7. LoopGain:Loop gain istheproduct ofbranchgainwhichtravels inthe loop.



ConstructionofSignalFlowGraph

Letusconstructasignalflowgraph byconsideringthefollowingalgebraicequations−

$y_2= a_{12}\, y_1 + a_{42}$

$y_4 y_3$

$=a_{23}y_2+a_{53}y_5y_4=a_3$

${}_4y_3$

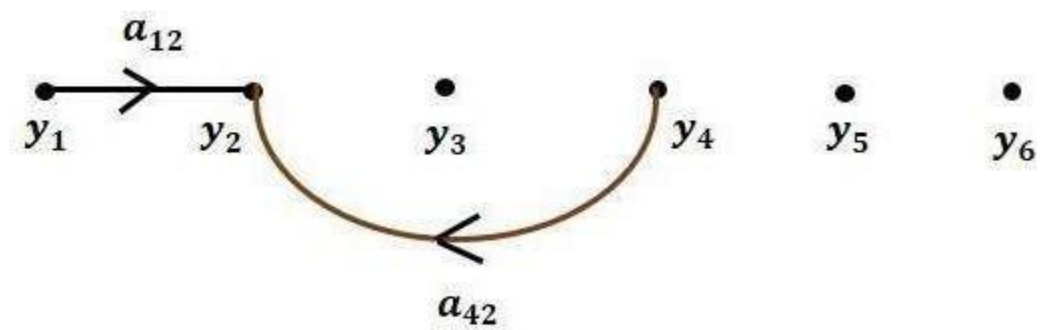
$y_5=a_{45}y_4+a_{35}y_3y_6$

$= a_{56}y_5$

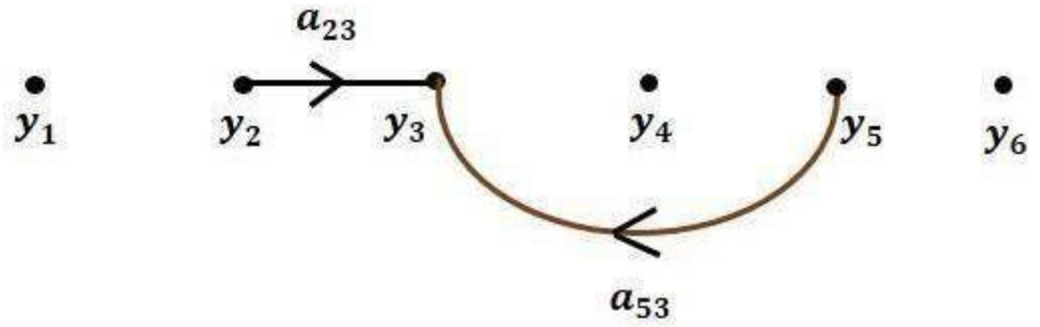
Therewillbesixnodes(y_1,y_2,y_3,y_4,$y_5$andy_6)andeightbranchesinthissignalflowgraph.Thegains ofthebranchesarea$_{12}$,a_{23},a_{34},a_{45},a_{56},a_{42},$a_{53}$and a_{35}.

Togettheoverallsignalflowgraph,drawthesignalflowgraphforeachequation,thencombineallthesesi gnalflowgraphsand thenfollowthestepsgivenbelow −

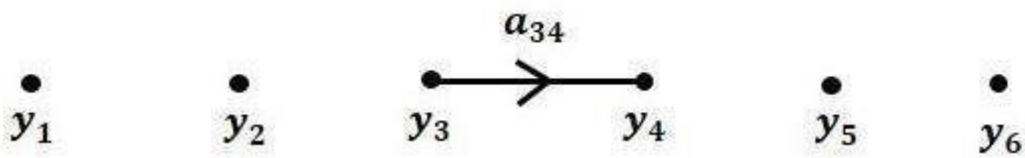
Step1−Signalflowgraphfor$y_2=a_{12}y_1+a_{42}y_4$isshowninthefollowingfigure.



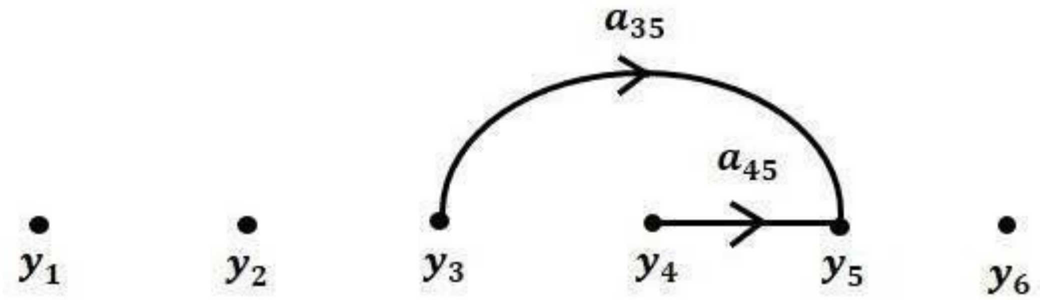
Step2−Signalflowgraphfor $y_3=a_{23}y_2+a_{53}y_5$isshowninthefollowingfigure.



Step3−Signalflowgraphfor$y_4=a_{34}y_3$isshowninthefollowingfigure.

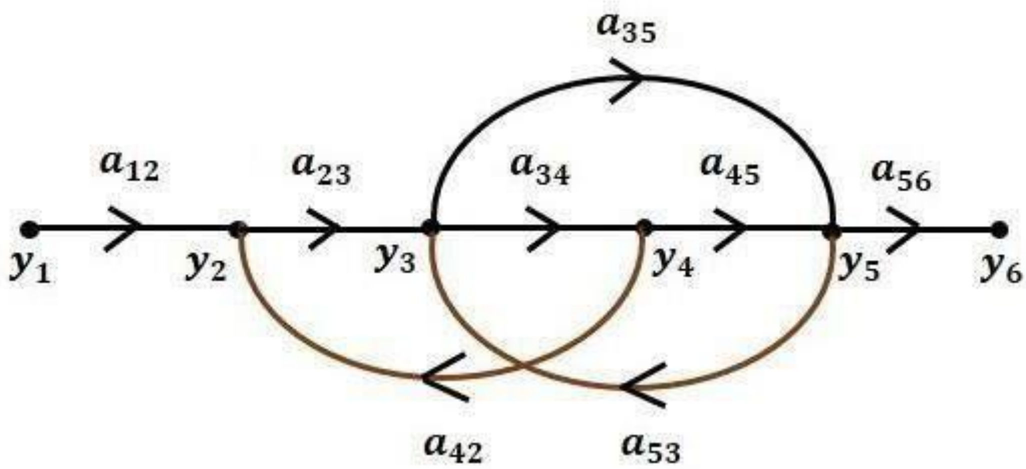


Step4−Signalflowgraphfor$y_5=a_{45}y_4+a_{35}y_3$isshowninthefollowingfigure.



Step5−Signalflowgraphfor$y_6=a_{56}y_5$isshowninthefollowingfigure.

Step6−Signalflowgraphofoverallsystemisshowninthefollowingfigure.



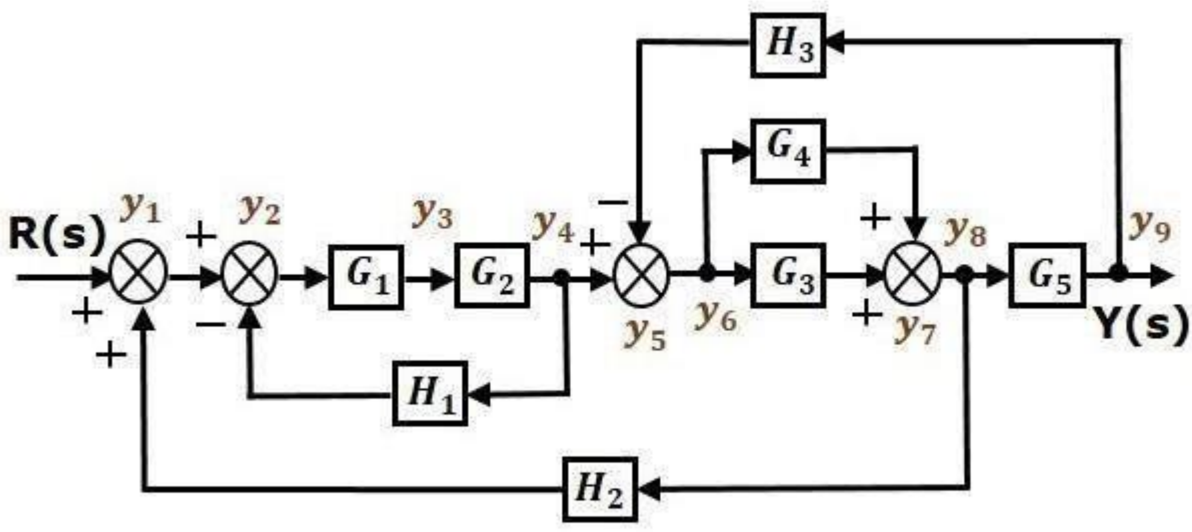
ConversionofBlockDiagramsintoSignalFlowGraphs

Followthesestepsforconvertingablockdiagramintoitsequivalent signalflowgraph.

- Represent all the signals, variables, summing points and take-off points of block diagramasnodesinsignalflowgraph.
- Representtheblocksofblockdiagramasbranchesinsignalflowgraph.
- Represent the transfer functions inside the blocks of block diagram as gains of thebranchesinsignalflowgraph.
- Connect the nodes as per the block diagram. If there is connection between two nodes(but there is no block in between), then represent the gain of the branch as one. Forexample, between summing points, between summing point and takeoff point, betweeninputandsummingpoint,betweentake-offpointandoutput.

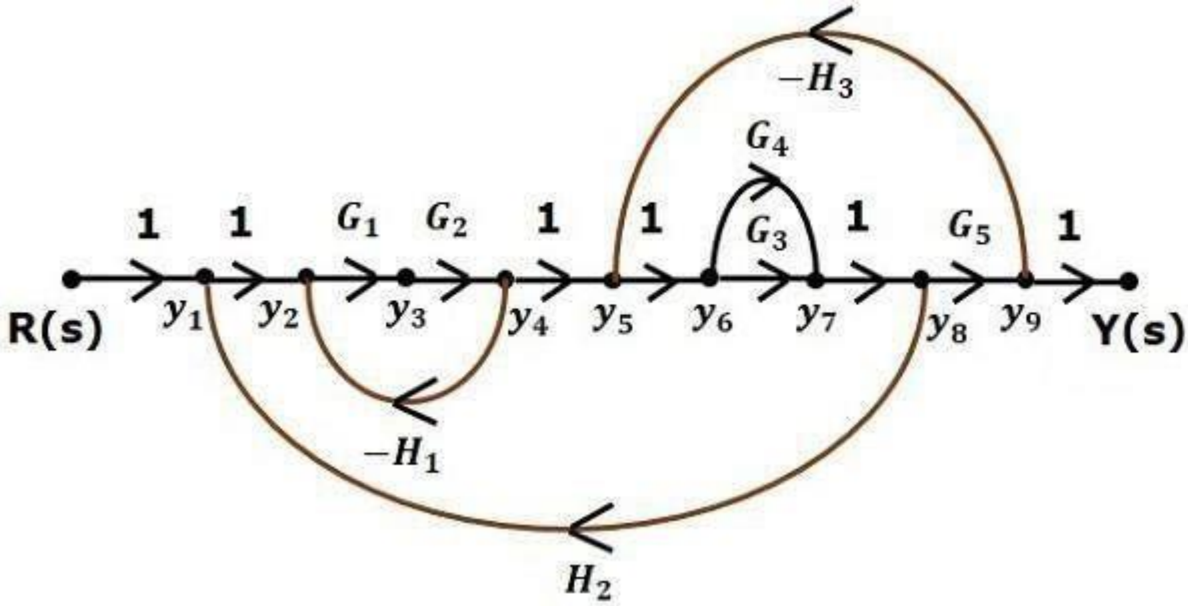
Example

Letusconvertthefollowingblockdiagramintoitsequivalent signalflowgraph.



Represent the input signal R(s) and output signal C(s) of block diagram as input node R(s) andoutputnodeC(s)ofsignalflowgraph.
Just for reference, the remaining nodes (y_1 to y_9) are labelled in the block diagram. There arenine nodes other than input and output nodes. That is four nodes for four summing points, fournodesforfourtake-offpoints and onenodeforthevariablebetweenblocksG1 and G2.

Thefollowing figureshowstheequivalentsignalflowgraph.

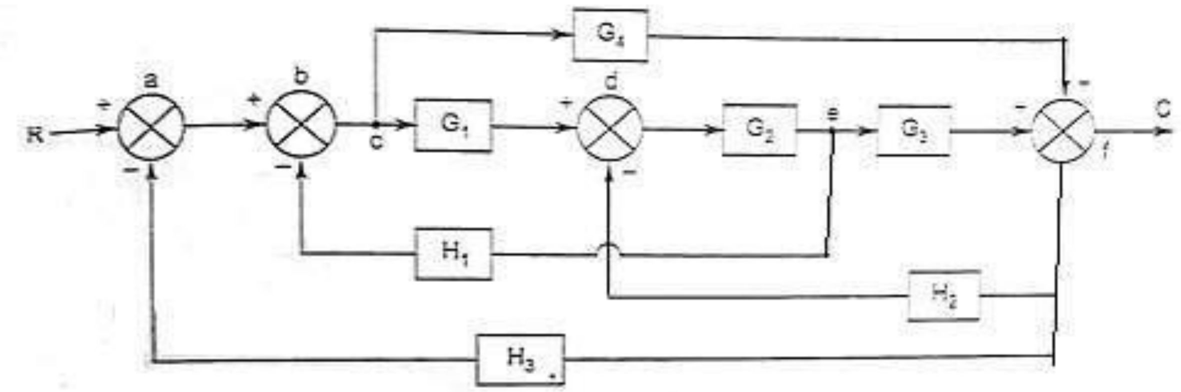


With the help of Mason’s gain formula (discussed in the next chapter), you can calculate thetransfer function of this signal flow graph. This is the advantage of signal flow graphs. Here, wenoneed tosimplify(reduce)thesignalflowgraphsfor calculating thetransfer function.

Note:1.Ifsummingpointispresentbeforeatakeoffpointitmaybeassumeassamenode.

2.If thereisapresentofsummingpointinseries(noblockwithinit),itmaybetakeiassamenode.

Ex:DeterminetransferfunctionbyusingMason’sgainformula.



Solution:

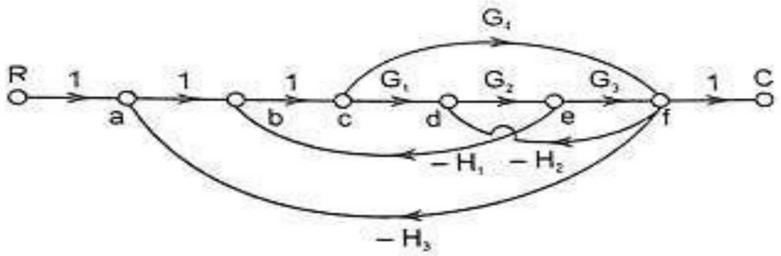


Fig. E3.13(a) *SFG*

Step I : Forward paths of the SFG are as follows :

(*i*) $a-b-c-d-e-f \quad \therefore \quad P_1 = G_1 G_2 G_3$

(*ii*) $a-b-c-f \quad \therefore \quad P_2 = G_4$

Step II : Individual loops of the SFG are as follows :

(*i*) $d-e-f-d \quad \therefore \quad L_1 = -G_2 G_3 H_2$

(*ii*) $b-c-d-e-b \quad \therefore \quad L_2 = -G_1 G_2 H_1$

(*iii*) $a-b-c-f-a \quad \therefore \quad L_3 = -G_4 H_3$

(*iv*) $b-c-f-d-e-b \quad \therefore \quad L_4 = G_4 H_2 G_2 H_1$

(*v*) $a-b-c-d-e-f-a \quad \therefore \quad L_5 = -G_1 G_2 G_3 H_3$

Step III : Gain products of all possible two non-touching loops are as follows :

Loop L_1 and Loop L_2 are non-touching loops

$$\therefore \qquad L_{12} = G_2 H_1 G_4 H_2$$

Step IV : $\qquad \Delta = 1 - (L_1 + L_2 + L_3 + L_4) + L_{12}$

Step V : For $\qquad P_1, \Delta_1 = 1$

$$\therefore \frac{C(s)}{R(s)} = \frac{G_1 G_2 G_3 G_4 G_5}{1 - \left(\begin{array}{c} G_2 H_1 + G_4 H_2 + G_2 G_3 G_4 G_5 G_6 G_7 G_8 \\ + G_2 G_3 G_4 G_5 G_6 G_8 H_4 \end{array} \right) + G_1 G_4 H_1 H_2} \text{. Ans.}$$

Mason'sGainFormula

Let us now discuss the Mason’s Gain Formula. Suppose there are ‘N’ forward paths in a signalflow graph. The gain between the input and the output nodes of a signal flow graph is nothingbutthetransferfunctionofthesystem.It canbecalculated byusingMason’sgainformula.

Mason’sgainformulais

$$T = C(s)/R(s) = \left(\frac{1}{\Delta}\right) \sum_{i=1}^{N} P_i \Delta_i$$

Where,

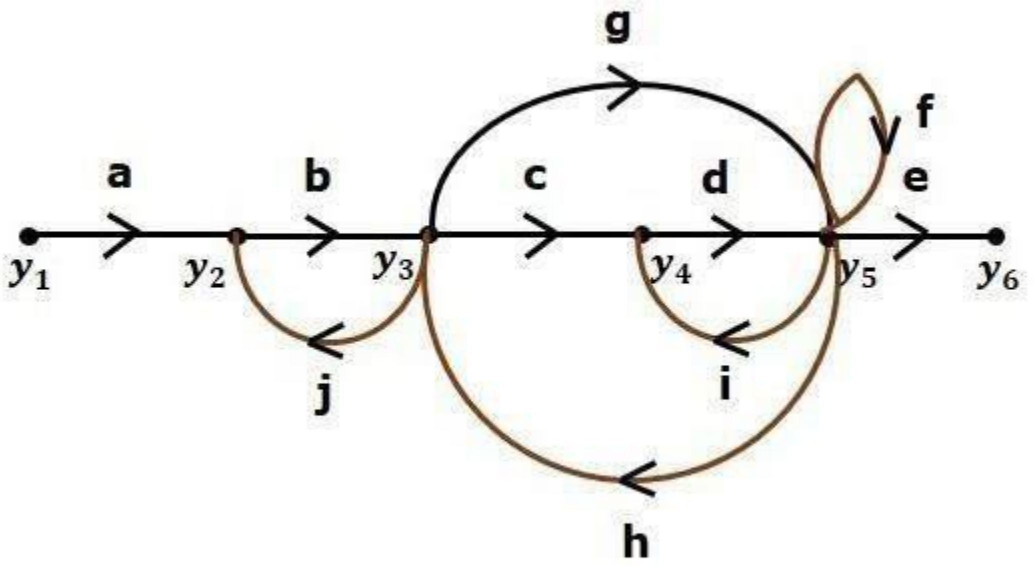
- C(s)istheoutput node
- R(s)istheinputnode

- TisthetransferfunctionorgainbetweenR(s)R(s)andC(s)C(s)
- $P_i$isthe i^{th} forwardpathgain

Δ=1−(sumofall individual loopgains)+ (sum ofgain productsofall possibletwonontouchingloops)−(sumofgainproductsofallpossiblethree nontouchingloops)+...

$\Delta_i$isobtainedfromΔbyremoving the loopswhicharetouchingthei^{th}forwardpath.

Considerthefollowingsignalflow graphin ordertounderstand thebasicterminologyinvolvedhere.



Path

Itisatraversalofbranchesfromonenodetoanyothernodeinthedirectionofbrancharrows.Itshouldnottraverseanynodemorethanonce.

Examplesy2→y3→y4→y5andy5→y3→y2Fo

rwardPath

Thepaththat existsfromtheinputnode tothe outputnodeis knownasforwardpath.

Examples−y1→y2→y3→y4→y5→y6andy1→y2→y3→y5→y6.Fo

rwardPathGain

Itisobtainedbycalculating theproductofallbranchgainsoftheforwardpath.

Examples–
abcdeistheforwardpathgainofy1→y2→y3→y4→y5→y6andabgeistheforwardpathgainofy1→y2→y3→y5→y6.

Loop

The path that starts from one node and ends at the same node is known as loop. Hence, it is aclosedpath.

Examples y2→y3→y2 and

y3→y5→y3.LoopGain

Itisobtainedbycalculatingtheproductofallbranchgainsofaloop.

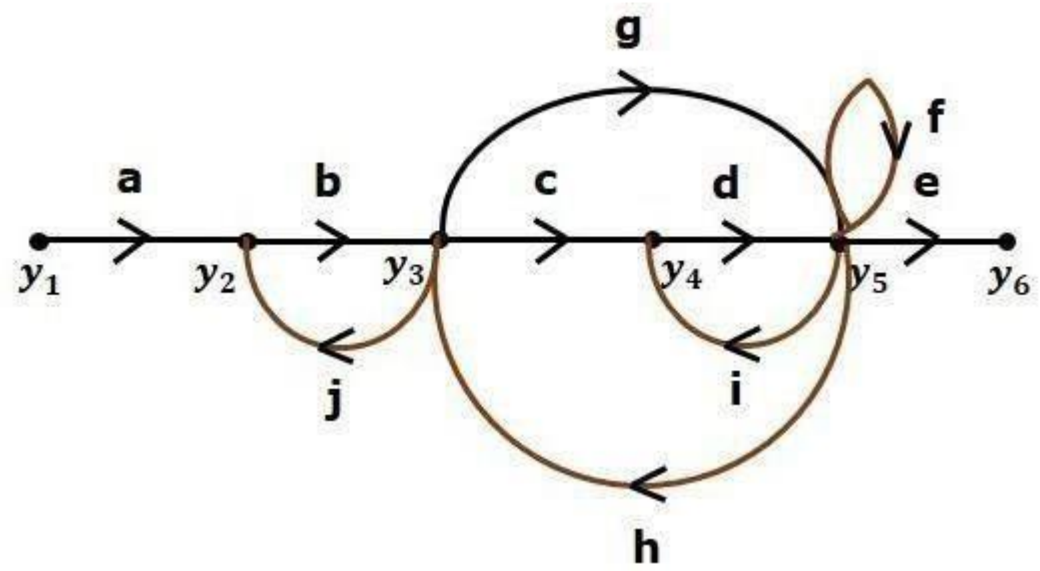
Examples−bjistheloop gainofy2→y3→y2 andghistheloopgainof y3→y5→y3.

Non-touchingLoops

These are the loops, which should not have any common node.Examples−Theloopsy2→y3→y2andy4→y5→y4arenon-touching.

CalculationofTransferFunctionusingMason's GainFormula

Letusconsiderthesamesignalflowgraphforfindingtransferfunction.



Numberofforward paths,N=2.

- Firstforwardpathis -y1→y2→y3→y4→y5→y6.
- Firstforwardpathgain,p1=abcde.
- Secondforwardpathis -y1→y2→y3→y5→y6.
- Second forward path gain,

p2=abge.Numberofindividualloops,

L=5.

Loops are - y2→y3→y2, y3→y5→y3, y3→y4→y5→y3 , y4→y5→y4 and y5→y5.Loopgainsare -l1=bj,l2=gh,l3=cdh,l4=diandl5=f

Numberoftwonon-touchingloops=2.

- Firstnon-touching loopspairis-y2→y3→y2, y4→y5→y4.
- Gainproductoffirstnon-touchingloopspair,l1l4=bjdi
- Secondnon-touchingloopspairis-y2→y3→y2,y5→y5
- Gainproduct ofsecondnon-touchingloopspairis-l1l5=bjf

Highernumberof(morethantwo)non-touchingloopsarenotpresentinthissignalflowgraph.Weknow,

Δ=1−(sumofallindividualloopgains)
+(sumofgainproductsofallpossibletwonontouchingloops)
−(sumofgainproductsofallpossiblethree nontouchingloops)+...

Substitutethevaluesintheaboveequation,Δ=1−(

bj+gh+cdh+di+f)+(bjdi+bjf)−(0)

$\Rightarrow \Delta = 1-(bj+gh+cdh+di+f)+bjdi+bjf$

Thereisnoloopwhichisnon-

touchingtothefirstforwardpath.So,$\Delta 1=1$
Similarly,$\Delta 2=1$.Since,noloopwhichisnon-touchingtothesecondforwardpath.Substitute,N=2 inMason'sgain formula

$$T=C(s)R(s)=[P1\Delta 1+P2\Delta 2]/\Delta$$

Substituteallthenecessaryvaluesintheaboveequation.

$$T=C(s)R(s)=(abcde)1+(abge)1/[1-(bj+gh+cdh+di+f)+bjdi+bjf]$$

$$\Rightarrow T=C(s)R(s)=(abcde)+(abge)/[1-(bj+gh+cdh+di+f)+bjdi+bjf]$$

Therefore,thetransferfunctionis-

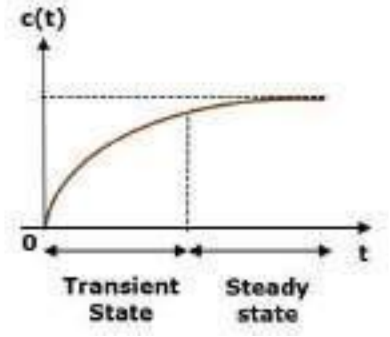
$$T=C(s)R(s)=(abcde)+(abge)/[1-(bj+gh+cdh+di+f)+bjdi+bjf]$$

Time ResponseAnalysis

Thevariationofoutputwithrespecttotimeisknownastimeresponse.Thetimeresponseconsists oftwoparts.

- Transientresponse
- Steadystateresponse

Here, both the transient and the steady statesare indicated in the figure 1. The responsescorresponding to these states are known astransientandsteadystateresponses.



Mathematically,wecanwritethetimeresponsec(t)as

$$C(t)=Ctr(t)+Css(t) \quad (1)$$

Figure1 Time response ofasystemWhere,

- c_{tr}(t)isthetransientresponse
- c_{ss}(t)isthesteadystateresponseTr

ansientResponse

Thetransientresponseisthepartofthetieresponsewhichgoestozeroafterlargeintervalof time ‘t’. Ideally, this value of ‘t’ is infinity and practically, it is five times

constant.Mathematically,wecanwriteitas

$$\lim_{t\to\infty} Ctr(t)=0$$

SteadystateResponse

The part of the time response that remains even after the transient response has zero value forlarge values of ‘t’ is known as steady stateresponse. This means, the transientresponse will bezeroevenduringthesteadystate.

Example

Letusfindthetransientandsteady state termsof thetimeresponseofthecontrolsystem c(t)=10+5e−t.Here, the second term 5e−t will be zero as t denotes infinity. So, this isthe transient term. Andthe first term 10 remains even as t approaches infinity. So, this is thesteadystateterm.

StandardTestSignals

The standard testsignalsareimpulse,step,rampandparabolic.Thesesignalsare usedtoknowtheperformanceofthecontrolsystemsusingtimeresponseofthe output.

Unit ImpulseSignal

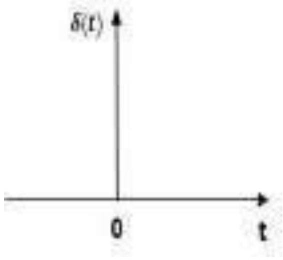
Asignalwhichhaszerovalueeverywhereexceptatt=0,whereitsmagnitudeisinfinite.Itisalsoknownas δ-function.Mathematically:

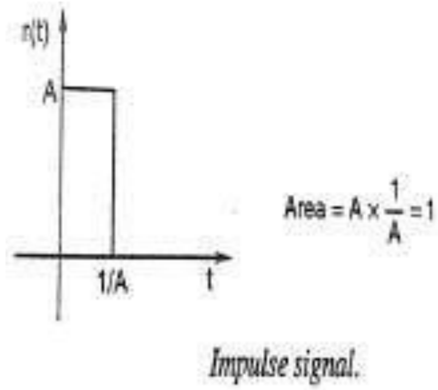
$$\delta(t)=0;t\neq0$$

$$=\infty;t=0 \quad (2)$$

and $\int_{-\epsilon}^{+\epsilon}\delta(t)dt = 1$ where ϵ tends to

zeroThefigure(2a)showsunitimpulsesignal.





Figure(2a) Figure(2b)

Practically a perfect impulse signal cannot be achieved. It is generally approximated be a pulseofunitareaas shown infigure(2b).

Anunitimpulsesignal isthederivativeofastepsignal i.e.,

$$\delta(t)=\frac{du(t)}{dt} \tag{3}$$

Laplacetransformofaunitimpulseis

$$L[\delta(t)]=L[\frac{du(t)}{dt}]=sR(s) = 1 \qquad \text{(AsforstepinputR(s)=1/s)} \tag{4}$$

<u>Unit StepSignal</u>

Aunitstepsignal isdefinedas r(t) =Au(t) (5)

Whereu(t)=1;t≥0

0;t<0

u(t)iscalledasunitstepsignal.

BytakingLaplace transformofr(t),wehave

R(s)=A/s (6)

Followingfigure3showsunitstepsignal.

So,theunitstepsignalexistsforallpositivevaluesof‘t’ includingzero

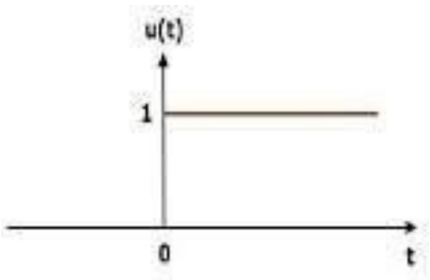


Figure3:unitstepsignal

UnitRamp Signal

Aunit rampsignal,r(t)isdefinedas

$$r(t)= At; t\geq 0$$

$$=0; t<0 \qquad (7)$$

Therampsignalstartsfromzeroandincreaseslinearlywithtime.Arampsignalistheintegralofastepsignal.i.e

Ramp Signal = ∫ step

signalThefigure4showsunitrampsignal

.

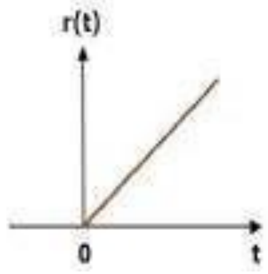


Figure4 : rampsignal

UnitParabolicSignal

Aunitparabolicsignal, r(t)isdefinedas,

$$r(t)=At^2/2; t\geq 0$$

$$0; t<0 \qquad (8)$$

By taking the Laplace transform of equation 8,$R(s)=A/s^3$ (9)

Parabolic signal is integral of a ramp signal.
i.eParabolicsignal=∫rampsignal

Thefigure5showstheunitparabolicsignal.

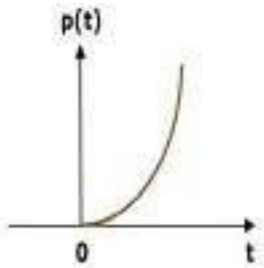
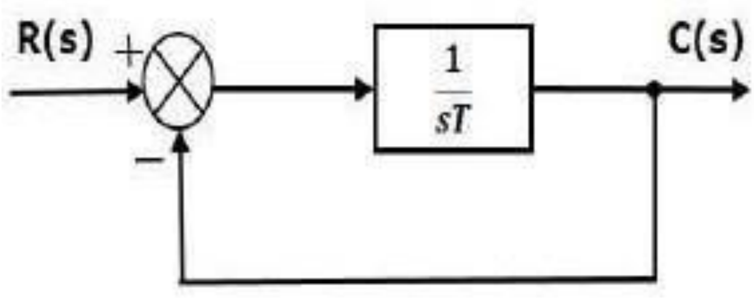


Figure5:unitparabolicsignal

So, the unit parabolic signal exists for all the positive values of ‘t’ including zero. And its valueincreases non-linearly with respect to ‘t’ during this interval. The value of the unit parabolicsignalis zerofor allthenegativevaluesof‘t’.

TimeResponseoftheFirst OrderSystem

Let us discuss the time response of the first order system. Consider the following block diagramof the closed loop control system. Here, an open loop transfer function, 1/sT is connected with aunitynegativefeedback.



Figure(6):Blockdiagramofafirst ordersystem

Weknowthatthetransferfunctionoftheclosedloopcontrolsystemhasunitynegativefeedbackas,

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

Substitute,$G(s)=\frac{1}{Ts}$ intheaboveequation.

$$\frac{C(s)}{R(s)} = \frac{\frac{1}{Ts}}{1+\frac{1}{Ts}} = \frac{1}{1+Ts} \qquad (10)$$

Thepowerofsisoneinthedenominatorterm.Hence,theabovetransferfunctionisofthefirstorder andthesystemis saidtobethefirstordersystem.

Wecanre-writetheaboveequationas

$$C(s) = \frac{1}{1+Ts}R(s) \qquad (11)$$

Where,

- C(s) is the Laplacetransformoftheoutput signalc(t),
- R(s)istheLaplace transformoftheinputsignalr(t),and
- Tisthetimeconstant.

<u>ImpulseResponseofFirstOrderSystem</u>For unitimpulsesignalR(s)=1

Considertheequation (11),$C(s) = \frac{1}{1+Ts}R(s)$

Substitute,R(s)=1 intheaboveequation.

$$C(s) = \frac{1}{1+Ts} \qquad (12)$$

Rearrangetheaboveequationinoneofthestandardformsof Laplacetransforms.

$$C(s) = \frac{\frac{1}{T}}{s+\frac{1}{T}} \qquad (13)$$

ApplyinverseLaplacetransformonbothsides.

$$c(t)=\frac{1}{T}e^{-\frac{t}{T}} \tag{14}$$

Theunitimpulseresponseisshowninthefigure7.Theunitimpulseresponse,c(t)isanexponentialdecayingsignalforpositivevaluesof't'anditiszerofornegativevaluesof't'.

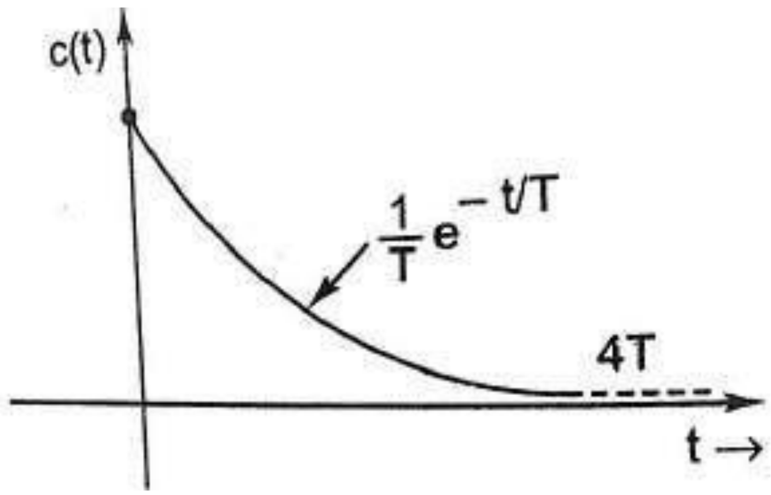


figure7:ImpulseResponseofFirstOrderSystem

StepResponseofFirstOrderSystemFor unitstepsignalR(s)=1/s

Considertheequation (11),$C(s) = \frac{1}{1+Ts}R(s)$

$$\therefore \quad C(s)=\frac{1}{1+Ts}\frac{1}{s}=\frac{1}{s(1+Ts)} \tag{15}$$

TakingPartialfractionsofEqu.(15)

$$C(s)=\frac{1}{s(1+Ts)}=\frac{A}{s}+\frac{B}{1+Ts} \tag{16}$$

$$\Rightarrow \frac{1}{s(1+Ts)}=\frac{A(1+Ts)+Bs}{s(1+Ts)}$$

$$\Rightarrow 1=A(sT+1)+Bs \tag{17}$$

BysolvingEqu.(17),weget

$$A=1;B=-T$$

Substitute,A=1and B=−Tin Equ.(16),weget

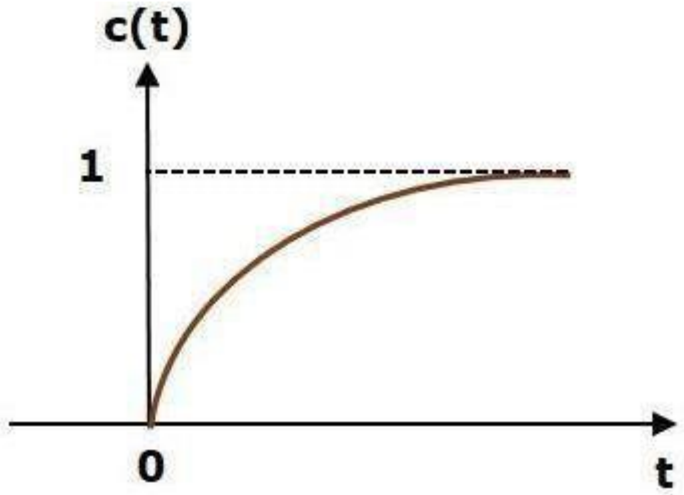
$$C(s)=\frac{1}{s}+\frac{(-T)}{1+Ts}$$

$$\Rightarrow C(s)=\frac{1}{s}-\frac{1}{\frac{1}{T}+s}$$

ApplyinverseLaplacetransformonboththe sides.

$$c(t)=1-e^{-t/T} \qquad (18)$$

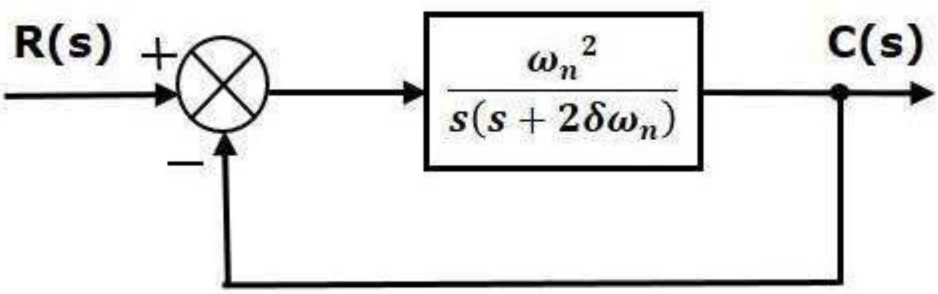
Thefollowing figureshowstheunitstepresponse.



The value of the unit step response, c(t) is zero at t = 0 and for all negative values of t. It isgradually increasing from zero value and finally reaches to one in steady state. So, the steadystatevaluedepends onthemagnitudeoftheinput.

TimeResponseofSecond OrderSystem

Considerthefollowingblockdiagramofclosedloopcontrolsystem.Here,anopenlooptransferfunction ,$\frac{\omega_n^2}{s(s+2\zeta\omega_n)}$isconnectedwith aunitynegativefeedback.



Weknowthatthetransferfunctionoftheclosedloopcontrolsystemhavingunitynegativefeedbackas

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

Substitute, $G(s)= \frac{\omega_n^2}{s(s+2\zeta\omega_n)}$ in the above equation.

$$C(s)/R(s)= \frac{\frac{\omega_n^2}{s(s+2\zeta\omega_n)}}{1+\frac{\omega_n^2}{s(s+2\zeta\omega_n)}} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \tag{19}$$

The power of 's' is two in the denominator term. Hence, the above transfer function is of the second order and the system is said to be the second order system.

The characteristic equation is-

$$s^2+2\zeta\omega_n s+\omega_n^2=0 \tag{20}$$

The roots of characteristic equation are -

$$S_1, S_2=[-2\omega_n\zeta \pm \sqrt{(2\zeta\omega_n)^2-4\omega_n^2}]/2$$

- The two roots are imaginary when $\zeta=0$.
- The two roots are real and equal when $\zeta=1$.
- The two roots are real but not equal when $\zeta>1$.
- The two roots are complex conjugate when $0 <\zeta <1$.

We can write C(s) equation as,

$$C(s)= \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} R(s) \tag{21}$$

Step Response of Second Order System

Consider the unit step signal as an input to the second order system. Laplace transform of the unit step signal is,

$$R(s)=1/s$$

We know the transfer function of the second order closed loop control system is,

$$C(s)/R(s)= \frac{m_n^2}{s^2+2\zeta m_n s+m_n^2}$$

Case 1: $\zeta=0$ (undamped system)

Substitute, $\delta=0$ in the transfer function.

$$C(s)/R(s)= \frac{\omega_n^2}{s^2+\omega_n^2}$$

$$\Rightarrow C(s)= \frac{\omega_n^2}{s^2+\omega_n^2} R(s) \tag{22}$$

Substitute, $R(s)=1/s$ in equation 22

$$C(s)=\frac{\omega_n^2}{s(s^2+\omega_n^2)} \quad (23)$$

Byusingpartialfraction the equation4canbewrittenas

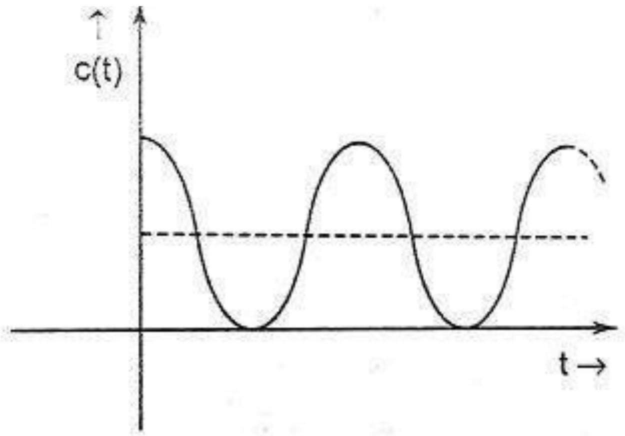
$$C(s)=\frac{A}{s}+\frac{Bs+C}{s^2+\omega_n^2} \quad (24)$$

After partialfractionA=1;B =-1;C=0 $\therefore C(s)=\frac{1}{s}-\frac{s}{s^2+\omega_n^2}$ (25)

ApplyinverseLaplacetransformonboththesides.

$$c(t)=1-\cos(\omega_n t) \quad (26)$$

So, the unit step response of the second order system when $\zeta = 0$ will be a continuous timesignal with constant amplitude and frequency. Since there is no damping with the time, thisresponse does not die out with time. This response is known as undamped response as shown inthefigure.



Undamped response of second order system with unit step input.

Case2: $\zeta=1$ (criticallydamped)

Substitute, $\zeta=1$ inthetransferfunction.

$$C(s)/R(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}=\frac{\omega_n^2}{s^2+2\omega_n s+\omega_n^2}$$

$$\Rightarrow C(s)=R(s)\,\frac{\omega_n^2}{[s+\omega_n]^2} \quad (27)$$

Substitute,R(s)=1/sinequation27

$$C(s)=\frac{\omega_n^2}{s[s+\omega_n]^2} \quad (28)$$

DopartialfractionsofEquation28

$$C(s)=\frac{\omega_n^2}{s[s+\omega_n]^2}=\frac{A}{s}+\frac{B}{s+\omega_n}+\frac{C}{[s+\omega_n]^2} \quad (29)$$

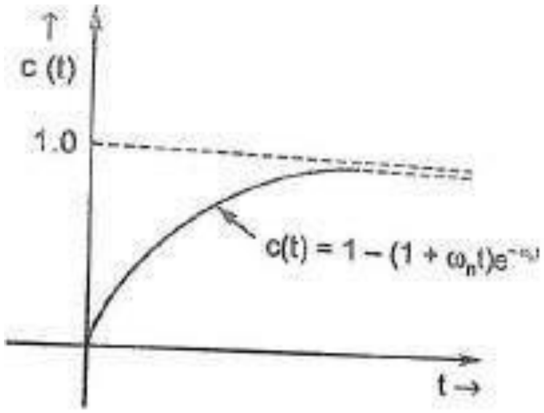
After simplifying, you will get the values of A, B and C as 1, −1 and and$-\omega_n$ respectively.Substitutethesevaluesintheabovepartialfractionexpansionof C(s).

$$C(s)=\frac{1}{s}+\frac{-1}{s+\omega_n}+\frac{-\omega_n}{[s+\omega_n]^2} \quad (30)$$

ApplyinverseLaplacetransformonboththesidesofEqs 30

$$c(t)=(1-e^{-\omega nt}-\omega_n t e^{-\omega nt})=1-e^{-\omega nt}(1+\omega_n t) \quad (31)$$

So,theunitstepresponseofthesecondordersystemwilltrytoreachthestepinputinsteadystate.



Critically damped system (ξ = 1).

Case3: 0 <ζ<1(underdamped system)

FromEquation(21) $C(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}R(s)$

Substitute,R(s)=1/s,Hence $C(s)=\frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)}$

Putting $s^2+2\zeta\omega_n s+\omega_n^2=[s+\zeta\omega_n]^2+\omega_n^2(1-\zeta^2)$,weget

$$C(s)=\frac{1}{s}\cdot\frac{\omega_n^2}{[s+\zeta\omega_n]^2+\omega_n^2(1-\zeta^2)} \quad (32)$$

Put$\omega_n^2(1-\zeta^2)=\omega_d^2$andby doingpartial fractionsof Equation32

$$C(s)=\frac{A}{s}+\frac{Bs+C}{[s+\zeta\omega_n]^2+\omega_d^2} \quad (33)$$

Afterpartialfractionsweget

A=1

B=-1

C =-2$\zeta\omega_n$

Putting the values of A, B,C in Equation 33,wehave

$$C(s)=\frac{1}{s}-\frac{s+2\zeta\omega_n}{[s+\zeta\omega_n]^2+\omega_d^2}$$

$$=\frac{1}{s}-\frac{s+\zeta\omega_n}{[s+\zeta\omega_n]^2+\omega_d^2}-\frac{\zeta}{\sqrt{1-\zeta^2}}\frac{\omega_n\sqrt{1-\zeta^2}}{[s+\zeta\omega_n]^2+\omega_d^2}$$

$$=\frac{1}{s}-\frac{s+\zeta\omega_n}{[s+\zeta\omega_n]^2+\omega_d^2}-\frac{\zeta}{\sqrt{1-\zeta^2}}\frac{\omega_d}{[s+\zeta\omega_n]^2+\omega_d^2} \qquad (34)$$

Taking inverse laplace transform of Equation 34

$$C(t)=1-e^{-\zeta\omega_n t}\cos(\omega_d t)-\frac{\zeta}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin(\omega_d t)$$

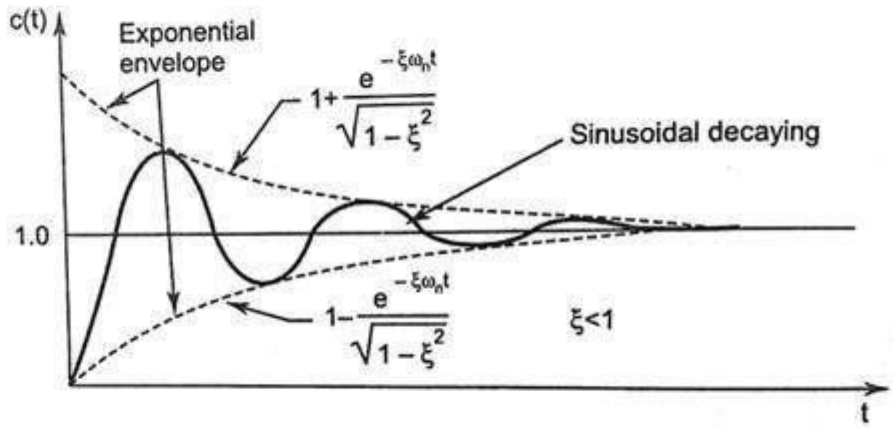
$$=1-\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\sqrt{1-\zeta^2}\cos(\omega_d t)-\zeta\sin(\omega_d t)\right] \qquad (35)$$

Now putting $\zeta=\cos\phi$; $\sqrt{1-\zeta^2}=\sin\phi$ Since $\phi=\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$, hence equation 35 becomes

$$C(t)=1-\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\sin\phi\cos(\omega_d t)-\cos\phi\sin(\omega_d t)\right]$$

$$=1-\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t+\phi) \qquad (36)$$

Equation (36) represents the solution for $0<\zeta<1$ and it is represented in figure as given below.



Underdamped response of second order system to unit step input.

Case 4: $\zeta>1$

We can modify the denominator term of the second order transfer function as follows−

$$s^2+2\zeta\omega_n s+\omega_n^2=[s+\zeta\omega_n]^2-\omega_n^2(\zeta^2-1)$$

Hence from equation (32)

$$C(s)=\frac{1}{s}\cdot\frac{\omega_n^2}{[s+\zeta\omega_n]^2-\omega_n^2(\zeta^2-1)}=\frac{1}{s}\cdot\frac{\omega_n^2}{(s+\zeta\omega_n+\omega_n\sqrt{\zeta^2-1})(s+\zeta\omega_n-\omega_n\sqrt{\zeta^2-1})} \qquad (37)$$

by doing partial fractions of Equation 37

$$C(s) = \frac{A}{s} + \frac{B}{(s+\zeta\omega_n+\omega_n\sqrt{\zeta^2-1})} + \frac{C}{(s+\zeta\omega_n-\omega_n\sqrt{\zeta^2-1})} \tag{38}$$

Aftersimplifying,youwillgetthevaluesof

$$A=1$$

$$B=\frac{1}{2}\frac{\zeta-\sqrt{\zeta^2-1}}{\sqrt{\zeta^2-1}}$$

$$C=-\frac{1}{2}\frac{\zeta+\sqrt{\zeta^2-1}}{\sqrt{\zeta^2-1}}$$

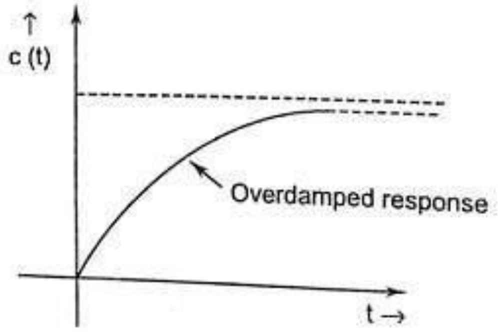
Substitutethe valueofA,B,Cinequation(38)

$$C(s) = \frac{1}{s} + \frac{1}{2}\frac{\zeta-\sqrt{\zeta^2-1}}{\sqrt{\zeta^2-1}}\frac{1}{(s+\zeta\omega_n+\omega_n\sqrt{\zeta^2-1})} - \frac{1}{2}\frac{\zeta+\sqrt{\zeta^2-1}}{\sqrt{\zeta^2-1}}\frac{1}{(s+\zeta\omega_n-\omega_n\sqrt{\zeta^2-1})} \tag{39}$$

ApplyinverseLaplacetransformofequation(29)wehave

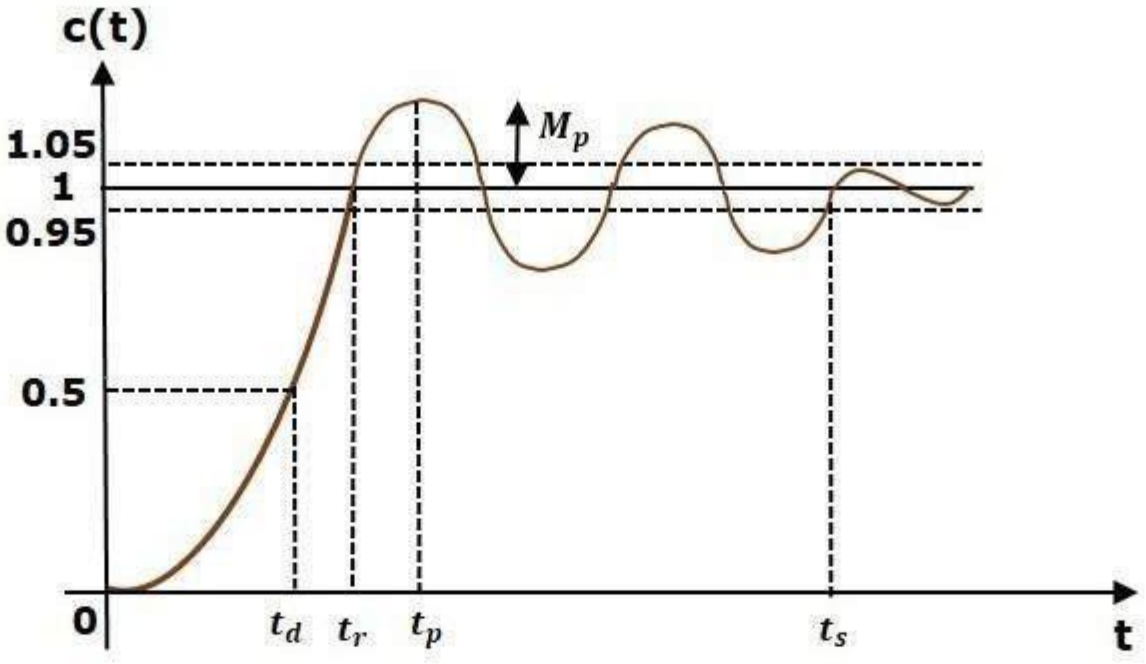
$$c(t) = 1 + \frac{\left(\xi-\sqrt{\xi^2-1}\right)}{2\sqrt{\xi^2-1}}e^{-\left(\xi\omega_n+\omega_n\sqrt{\xi^2-1}\right)t} - \frac{\xi+\sqrt{\xi^2-1}}{2\sqrt{\xi^2-1}}.e^{-\left(\xi\omega_n-\omega_n\sqrt{\xi^2-1}\right)t} \tag{40}$$

Equation(40)representsthesolution for$\zeta>1$ anditisrepresentedinfigureasgivenbelow.



TimeDomainSpecifications

Let us discuss the time domain specifications of the second order system. The step response ofthesecond ordersystemfortheunderdamped caseisshowninthefollowingfigure.



All the time domain specifications are represented in this figure. The respse up to the settlingtime is known as transient response and the response after the settling time is known as steadystateresponse.

1. DelayTime

Itisthetimerequiredfortheresponsetoreach50%ofitsfinalvalueinfirstattempt.Itisdenotedbyt_d.

Considerthestepresponseofthesecondordersystemfort≥0,when'ζ'liesbetweenzeroandone.Fromequation(36)

$$c(t)=1-\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t+\phi)$$

The finalvalueofthestepresponseisone.

Therefore,at $t=t_d$,thevalueofthestepresponsewillbe0.5.Substitutethesevaluesintheaboveequation.

$$c(t_d)=0.5=1-\frac{e^{-\zeta\omega_n t_d}}{\sqrt{1-\zeta^2}}\sin(\omega_d t_d+\phi)$$

$$\Rightarrow \frac{e^{-\zeta\omega_n t_d}}{\sqrt{1-\zeta^2}}\sin(\omega_d t_d+\phi)=0.5$$

Byusinglinearapproximation,you willgetthedelaytime$t_d$as

$$t_d=(1+0.7\zeta)/\omega_n \tag{40}$$

2. RiseTime

Itisthetimerequiredfortheresponsetorisefrom10%to90%ofthefinalvalueforoverdamped system and 0% to 100% of its final value for the under-damped systems. Rise timeisdenotedbyt_r.

Att=t_r,c(t)=1

Hencefromequation(36)

$$c(t)=1-\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d \mathbf{t}+\phi)$$

$$c(t_r)=1=1-\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t_r+\phi)$$

$$\Rightarrow\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t_r+\phi)=0$$

$$\Rightarrow\sin(\omega_d t_r+\phi)=0$$

$$\Rightarrow\omega_d t_r+\phi=\pi$$

$$\Rightarrow t_r=(\pi-\phi)/\omega_d \tag{41}$$

From aboveequation,we can conclude thattherise time tr andthe dampedfrequency $\omega_d$areinverselyproportionaltoeachother.

3. PeakTime

Itisthetimerequiredfortheresponsetoreachthepeakvalueforthefirsttime.Itisdenotedbyt_p.

Weknowthestepresponseofsecondordersystemforunder-dampedcaseis(fromequation36)

$$c(t)=1-\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d \mathbf{t}+\phi)$$

Att=t_p,thefirstderivateoftheresponseiszero. Hence

$$\frac{dc(t_p)}{dt}=0$$

$$\Rightarrow 0-\frac{e^{-\zeta\omega_n t_p}(-\zeta\omega_n)sin(\omega_d t_p+\ \phi)}{\sqrt{1-\zeta^2}}-\frac{e^{-\zeta\omega_n t_p}\ \omega_d\cos(\omega_d t_p+\ \phi)}{\sqrt{1-\zeta^2}}=0$$

$$\Rightarrow \frac{e^{-\zeta\omega_n t_p}(-\zeta\omega_n)sin(\omega_d t_p+\phi)}{\sqrt{1-\zeta^2}}+\frac{e^{-\zeta\omega_n t_p}\omega_d\cos(\omega_d t_p+\phi)}{\sqrt{1-\zeta^2}}=0$$

$$\Rightarrow\zeta\omega_n sin(\omega_d t_p+\phi)=\omega_d\cos(\omega_d t_p+\phi)$$

$$\Rightarrow\tan(\omega_d t_p+\phi)=\omega_d/\zeta\omega_n$$

By putting$\omega_d=\omega_n\sqrt{1-\zeta^2}and\phi=\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$

$$\tan(\omega_n\sqrt{1-\zeta^2}t_p+\phi)=\frac{\omega_n\sqrt{1-\zeta^2}}{\zeta\omega_n}=\tan\phi$$

or $\omega_n\sqrt{1-\zeta^2}t_p+\phi=n\pi+\phi$forn =1

$$t_p=\frac{\pi}{\omega_n\sqrt{1-\zeta^2}} \tag{42}$$

4. PeakOvershoot

PeakovershootM$_p$isdefinedasthedeviationoftheresponseatpeaktimefromthefinalvalueofresponse.It isalsocalled themaximumovershoot.

Mathematically,wecanwriteit as

$$M_p = c(t_p) - c(\infty)$$

Where,

$c(t_p)$isthepeak valueoftheresponse.

$c(\infty)$ isthefinal(steadystate)valueoftheresponse.

Fromequation(36) $c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \phi)$

Putt=$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$ and $\omega_d = \omega_n\sqrt{1-\zeta^2}$

∴ $$c(t_p) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_n\sqrt{1-\zeta^2}\, t_p + \phi)$$

$$= 1 - \frac{e^{-\zeta\omega_n \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}} \sin\left(\omega_n\sqrt{1-\zeta^2}\,\frac{\pi}{\omega_n\sqrt{1-\zeta^2}} + \phi\right)$$

$$= 1 - \frac{e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}} \sin(\pi + \phi) = 1 + \frac{e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}} \sin\phi$$

$$= 1 + \frac{e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}} \sqrt{1-\zeta^2} \qquad \text{since} \sin\phi = \sqrt{1-\zeta^2}$$

∴ $$c(t_p) = 1 + e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}$$

∴ $$\%M_P = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100 = \frac{1 + e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} - 1}{1} \times 100$$

Or $$\%M_P = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100 \qquad (43)$$

5. Settlingtime

It is the time required for the response to reach the steady state and stay within the specifiedtolerance bands around the final value. In general, the tolerance bands are 2% and 5%. Thesettlingtimeis denotedbyt_s.

Asseenfromequation26thetimeconstantoftheexponentialenvelopeis$T = 1/\zeta\omega_n$.

The settling time of the second order system for 2% tolerance band is appx. Four times the timeconstantTi.e.

$$t_s = 4/$$

$\zeta\omega_n$=4TThesettling timefor5%tolerancebandis-

$t_s = 3/\zeta\omega_n = 3T$

Steady State Error

It indicate the error between the actual output and the desired

output as t tends to infinity i.e. $e_{ss} = \lim_{t\to 0}[r(t) - c(t)] = \lim_{t\to 0} r(t) - \lim_{t\to 0} c(t)$

$$= \lim_{t\to 0} 1 - \lim_{t\to 0}\left[1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d \mathbf{t} + \phi)\right] = 1 - 1 = 0$$

Thus second order system has zero steady state error to unit step input.

Example

Let us now find the time domain specifications of a control system having the closed loop transfer function $4s^2 + 2s + 4$ when the unit step signal is applied as an input to this control system.

We know that the standard form of the transfer function of the second order closed loop control system as

$C(s)/R(s) = \omega_n^2 / s^2 + 2\zeta\omega_n s + \omega_n^2$

By equating these two transfer functions, we will get the un-damped natural frequency ω_n as 2 rad/sec and the damping ratio ζ as 0.5.
We know the formula for damped frequency ω_d as

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

Substitute ω_n and δ values in the above formula.

$$\Rightarrow \omega_d = 2\sqrt{1-0.5^2}$$

$$\Rightarrow \omega_d = 1.732 \text{ rad/sec}$$

Substitute, δ value in following relation

$$\Phi = \cos^{-1}\zeta$$

$$\Rightarrow \phi = \cos^{-1}(0.5) = \pi/3 \text{ rad}$$

Substitute the above necessary values in the formula of each time domain specification and simplify in order to get the values of time domain specifications for given transfer function.

SteadyStateErrorAnalysis

Thedeviationoftheoutputofcontrolsystemfromdesiredresponseduringsteadystateisknown as steady state error. It is represented as e_{ss}. We can find steady state error using thefinalvaluetheoremasfollows.

$$e_{ss}=\lim_{t\to\infty}e(t)=\lim_{s\to 0}sE(s)$$

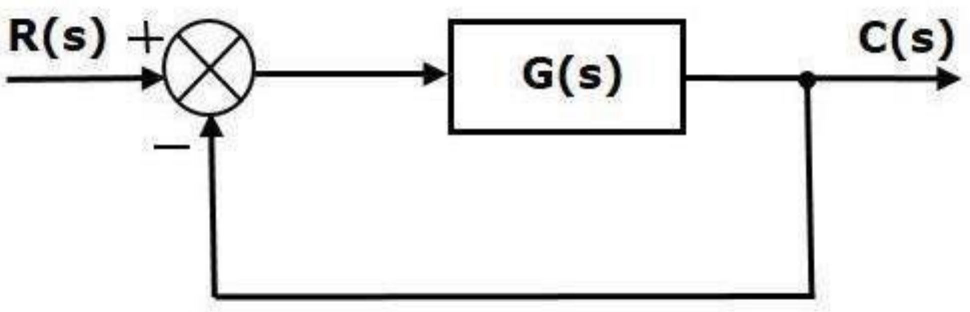
Where,

E(s)istheLaplace transformoftheerrorsignal,e(t)

Letusdiscusshowtofindsteadystateerrorsforunityfeedbackandnon-unityfeedbackcontrolsystemsonebyone.

SteadyStateErrorsforUnityFeedbackSystems

Considerthe followingblock diagram of closedloopcontrol system,whichishavingunitynegativefeedback.



Where,

- R(s)istheLaplace transformofthe reference Input signalr(t)
- C(s)istheLaplacetransformofthe outputsignal c(t)

Weknowthetransferfunctionoftheunitynegativefeedbackclosedloopcontrolsystemas

$$C(s)/R(s)=G(s)/1+G(s)$$

$$\Rightarrow C(s)=R(s)G(s)/1+G(s)$$

Theoutputofthesumming pointis-

$$E(s)=R(s)-B(s)=R(s)-C(s)H(s)=R(s)-E(s)G(s)H(s)$$

$$\Rightarrow E(s)[1+G(s)H(s)]=R(s)$$

$$\Rightarrow E(s)=R(s)/[1+G(s)H(s)]$$

For H(s)=1 $\quad E(s)=R(s)/[1+G(s)]$

SubstituteE(s)valueinthesteadystateerror formula

$$e_{ss} = \lim_{s \to 0} sR(s)/1+G(s)$$

Thefollowingtableshowsthesteadystateerrorsandtheerrorconstantsforstandardinputsignalslikeunit step,unitramp &unitparabolicsignals.

Inputsignal	Steadystateerrore_{ss}	Errorconstant
unitstepsignal	1/1+kp	$K_p=\lim_{s\to 0}G(s)$
unit rampsignal	1/Kv	$K_v=\lim_{s\to 0}sG(s)$
unit parabolicsignal	1/Ka	$K_a=\lim_{s\to 0}s2G(s)$

WhereKp,KvandKaarepositionerrorconstant,velocityerrorconstantandaccelerationerrorconstantr espectively.

Note –

1. Ifanyoftheaboveinputsignalshastheamplitudeotherthanunity,thenmultiplycorrespondingstead ystateerrorwiththatamplitude.

2. We can't define the steady state error for the unit impulse signal because, it exists only atorigin. So, we can't compare the impulse response with the unit impulse input as t denotesinfinity.

Example

Let us find the steady state error for an input signal $r(t)=(5+2t+t^2/2)u(t)$ of unity negativefeedbackcontrolsystemwith$G(s)=5(s+4)/s^2(s+1)(s+20)$

The given input signal is a combination of three signals step, ramp and parabolic. The followingtableshowsthe errorconstantsand steadystateerrorvaluesforthesethreesignals.

Inputsignal	Errorconstant	Steadystateerror
$r_1(t)=5u(t)$	$K_p=\lim_{s\to 0}G(s)=\infty$	$e_{ss1}=5/1+kp=0$
$r_2(t)=2tu(t)$	$K_v=\lim_{s\to 0}sG(s)=\infty$	$e_{ss2}=2/Kv=0$
$r_3(t)=t^2/2u(t)$	$K_a=\lim_{s\to 0}s^2G(s)=1$	$e_{ss3}=1/ka=1$

Wewillgettheoverallsteadystate error,byaddingtheabove threesteadystateerrors.

$$e_{ss}=e_{ss1}+e_{ss2}+e_{ss3}$$

$$\Rightarrow e_{ss}=0+0+1=1$$

Therefore,wegotthesteadystateerrore_{ss}as1forthisexample.

Steadystateerrorfordifferenttypesofinputfor Type-0, Type-1andType-2 systems.

Input	'Type-0' system			'Type-1' system		'Type-2' system	
	$\mathcal{L}$ (input)	Static error coefficient	steady state error e_{ss}	Static error coefficient	Steady state error e_{ss}	Static error coefficient	e_{ss}
Step input	$\frac{A}{s}$	$K_p = K$	$\frac{A}{1+K}$	$K_p = \infty$	0	$K_p = \infty$	0
Ramp input	$\frac{A}{s^2}$	$K_v = 0$	∞	$K_v = K$	$\frac{1}{K}$	$K_v = \infty$	0
Parabolic input	$\frac{A}{s^3}$	$K_a = 0$	∞	$K_a = 0$	∞	$K_v = K$	$\frac{1}{K}$

Effectofaddingpolesandzerototransferfunction

Effectofadditionofpoletotransferfunction:

1) As the pole moves towards the origin in s plane, the rise time increases and the maximumovershoot decreases, as far as the overshoot is concerned, adding a pole to the closed looptransfer function has just the opposite effect to that of adding a pole to forward path transferfunctionas discussedinthelastarticle.

2) Theadditionoflefthalfpoletendstoslowdownthesystemresponse.

3) The effectofadditionofpolebecomesmorepronouncedaspolelocation driftsaway fromimaginaryaxis.

4) Additionofrighthalfpolewillmakeoverallsystemresponseto

beanunstableone.Effectofadditionofzerototransferfunction:

1) Makesthesystemoverallresponsefaster.

2) Risetime,peaktime,decreasesbutovershootincreases.

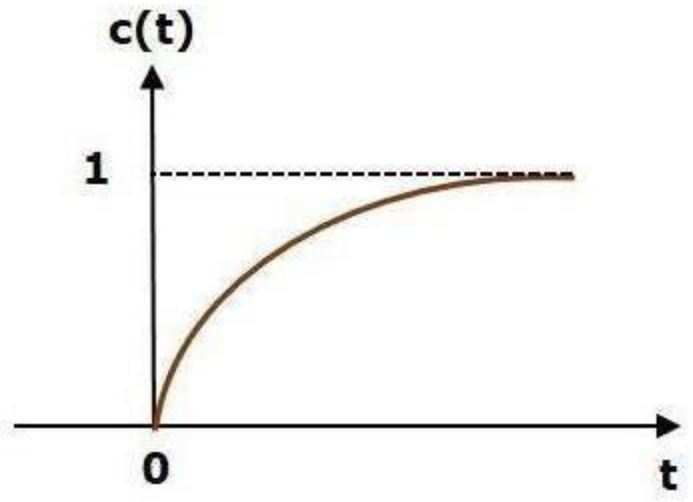
3) Addition of right half zeros means system response slower and system exhibits inverseresponse. Suchsystemsaresaidtobenon-minimumphasesystems.

Analysisof stabilitybyRootLocusTechnique

Stability:

Asystemissaidtobestable,ifitsoutputisundercontrol.Otherwise,itissaidtobeunstable.Astablesystem producesabounded output for agivenboundedinput.

Thefollowingfigureshowstheresponseofastablesystem.



This is the response of first order control system for unit step input. This response has thevalues between 0 and 1. So, it is bounded output. We know that the unit step signal has thevalue of one for all positive values of t including zero. So, it is bounded input. Therefore, thefirstordercontrolsystemis stablesinceboththe input and the outputarebounded.

TypesofSystems basedonStability

Wecanclassifythesystemsbased onstabilityasfollows.

- Absolutelystablesystem
- Conditionallystablesystem
- Marginallystablesystem

AbsolutelyStableSystem

If the system is stable for all the rangeof system componentvalues,thenitis known astheabsolutelystablesystem.Theopenloopcontrolsystemisabsolutelystableifallthepoles of the open loop transfer function present in left half of ‘s’ plane. Similarly, the closedloop control system is absolutely stable if all the poles of the closed loop transfer functionpresentinthelefthalfofthe‘s’plane.

ConditionallyStableSystem

Ifthesystemisstableforacertainrangeofsystemcomponentvalues,thenitisknownasconditionallys table system.

MarginallyStableSystem

If the system is stable by producing an output signal with constant amplitude and constantfrequency of oscillations for bounded input, then it is known asmarginally stable system.The open loop control system is marginally stable if any two poles of the open loop transferfunctionispresentontheimaginaryaxis.Similarly,theclosedloopcontrolsystemismarginally stable if any two poles of the closed loop transfer function is present on theimaginaryaxis.

Conditionforstability

Letusconsideratransferfunctionofaclosedloopsystem:

$$\frac{C(s)}{R(s)}=\frac{a_0s^m+a_1s^{m-1}+a_2s^{m-2}+...+a_{m-1}s^1+a_ms^0}{a_0s^n+a_1s^{n-1}+a_2s^{n-2}+...+a_{n-1}s^1+a_ns^0};$$

Here the characteristics Equation : $a_0s^n+a_1s^{n-1}+a_2s^{n-2}+...+a_{n-1}s^1+a_ns^0=0$

Necessaryandsufficientconditionsforstability:

1. Allthecoefficientsofthech.Equationshouldhavesamesign.
2. Thereshouldbeno missingterm.

Routh-HurwitzStabilityCriterion

ThiscriterionisbasedonorderingthecoefficientsofthecharacteristicsequationintoanarraycalledRouth's array. TheRouth'sarrayis formedas follows.

FollowthisprocedureforformingtheRouthtable.

- Fill the first two rows of the Routh array with the coefficients of the characteristicpolynomial as mentioned in the table. Start with the coefficient of s_n and continue uptothecoefficientofs_0.
- Fill the remaining rows of the Routh array with theelementsas mentioned in thetable . Continue this process till you get the first column elementof row $s_0$is a_n.Here,$a_n$isthe coefficientof$s_0$inthecharacteristicpolynomial.

Note−IfanyrowelementsoftheRouthtablehavesomecommonfactor,thenyoucandividetherowelementswiththatfactorforthesimplificationwillbeeasy.

Consider the characteristic equation of the order 'n' is -

$a_0s^n+a_1s^{n-1}+a_2s^{n-2}+...+a_{n-1}s^1+a_ns^0=0$

s^n	a_0	a_2	a_4	a_6	...	...
s^{n-1}	a_1	a_3	a_5	a_7	...	...
s^{n-2}	b1=(a1a2−a_3a0)/a1	b2=(a1a4−a5a0)/a1	b3=(a1a6−a7a0)/a1	...	...	...
s^{n-3}	c1=(b1a3−b2a1)/b1	c2=(b1a5−b3a1)/b1	⋮⋮			

⁝	⁝	⁝	⁝			
S^1	⁝	⁝				
S^0	an					

SufficientConditionforRouth-HurwitzStability

The sufficient condition is that all the elements of the first column of the Routh array shouldhave the same sign. This means that all the elements of the first column of the Routh arrayshouldbeeitherpositiveornegative.

Example

Letusfindthestabilityofthecontrolsystemhavingcharacteristic

equation,$S^4+3s^3+3s^2+2s+1=0$

Step1−VerifythenecessaryconditionfortheRouth-Hurwitzstability.

Allthecoefficientsofthecharacteristicpolynomial,$S^4+3s^3+3s^2+2s+1$arepositive.So,thecontrolsystemsatisfies the necessarycondition.

Step2−Formthe Routharrayforthegivencharacteristicpolynomial.

S^4	1	3	1
S^3	3	2	
S^2	$\frac{(3\times3)-(2\times1)}{3}=7/3$	$\frac{(3\times1)-(0\times1)}{3}=1$	
S^1	$\frac{(7/3\times2)-(1\times3)}{7/3}=5/7$		
S^0	1		

Step3−VerifythesufficientconditionfortheRouth-Hurwitzstability.

AlltheelementsofthefirstcolumnoftheRoutharrayarepositive.ThereisnosignchangeinthefirstcolumnoftheRoutharray.So,thecontrolsystemis stable.

SpecialCasesofRouthArrayThet

wospecialcasesare−

- Thefirstelementofanyrowofthe Routharrayiszero.
- Allthe elementsofany rowofthe Routharrayarezero.

Letusnowdiscuss howto overcomethedifficultyinthesetwo cases,onebyone.

FirstElementofanyrowoftheRoutharrayiszero

IfanyrowoftheRoutharraycontainsonlythefirstelementaszeroandatleastoneoftheremainingelements havenon-zerovalue,thenreplacethefirstelementwithasmallpositive

integer,
ϵ.AndthencontinuetheprocessofcompletingtheRouthtable.Now,findthenumberofsignchangesinth e firstcolumnoftheRouthtablebysubstitutingϵtendstozero.

Example

Let us find the stability of the control system having characteristic equation,$S^4+2s^3+s^2+2s+1=0$

Step1−VerifythenecessaryconditionfortheRouth-Hurwitzstability.

Allthecoefficientsofthecharacteristicpolynomial,$S^4+2s^3+s^2+2s+1$arepositive.So,thecontrolsystem satisfiedthenecessarycondition.

Step2−FormtheRoutharrayforthegivencharacteristicpolynomial.

S^4	1	1	1
S^3	2	2	
S^2	0	1	
S^1			
S^0			

Special case(i) − Only thefirstelement of row s^2 is zero. So, replaceitby ϵ and continuetheprocess ofcompletingtheRouthtable
.

s4	1	1	1
s3	1	1	
s2	ϵ	1	
s1	$[(\epsilon\times1)-(1\times1]) / \epsilon = (\epsilon-1)/\epsilon$		
s0	1		

Step3−VerifythesufficientconditionfortheRouth-Hurwitzstability.Asϵ

tends to zero,the Routhtablebecomeslikethis.

s4	1	1	1
s3	1	1	
s2	0	1	

s1	-∞		
s0	1		

Therearetwosignchangesinthefirstcolumnof Routhtable.Hence,thecontrolsystemisunstable.

Allthe Elementsofanyrowofthe

Routharrayarezero Inthiscase,followthesetwosteps−

- Writetheauxilaryequation,A(s)oftherow,whichisjustabovetherowofzeros.
- Differentiatetheauxiliaryequation,A(s)withrespecttos.Filltherowofzeroswiththesecoef ficients.

Example

Letusfindthestabilityofthecontrolsystemhavingcharacteristicequation,$S^5+3s^4+s^3+3s^2+s+3=0$

Step1−Verifythenecessaryconditionforthe Routh-Hurwitzstability.

Allthecoefficientsofthegivencharacteristicpolynomialarepositive.
So,thecontrolsystemsatisfiedthe necessarycondition.

Step2−Formthe Routharrayforthegivencharacteristicpolynomial.

s5	1	1	1
s4	~~3~~1	~~3~~1	~~3~~1
s3	0	0	
s2			
s1			
s0			

The row s4 elements have the common factor of 3. So, all these elements are divided by 3.Specialcase(ii)−Alltheelementsofrow s3arezero.So,writetheauxiliaryequation,A(s)oftherows4.

$A(s)=s^4+s^2+1$

Differentiatetheaboveequationwithrespect tos.

$\frac{dA(s)}{ds}$=4s3+2s

Placethesecoefficientsinrows^3.

s5	1	1	1

s4	1	1	1
s3	4	2	
s2	0.5	1	
s1	−3		
s0	1		

Step3−VerifythesufficientconditionfortheRouth-Hurwitzstability.

There are two sign changes in the first column of Routh table. Hence, the control system isunstable.

In the Routh-Hurwitz stability criterion, we can know whether the closed loop poles areinon left half of the 's' plane or on the right half of the 's' plane or on an imaginary axis. So,we can't find the nature of the control system. To overcome this limitation, there is atechniqueknownastherootlocus.Wewilldiscussthistechniqueinthenext twochapters.

RootLocus

In the root locus diagram, we can observe the path of the closed loop poles. Hence, we canidentify the nature of the control system. In this technique, we will use an open loop transferfunctiontoknowthe stabilityoftheclosedloopcontrolsystem.

TheRootlocusisthelocusof therootsof thecharacteristicequationbyvaryingsystemgainKfromzerotoinfinity.

AngleConditionandMagnitudeCondition

The points on the root locus branches satisfy the angle condition. So, the angle condition isused to know whether the point exist on root locus branch or not. We can find the value of Kfor the points on the root locus branches by using magnitude condition. So, we can use themagnitudeconditionfor thepoints,andthissatisfiesthe anglecondition.

Characteristic equation of closed loop control system

is$1+G(s)H(s)=0$

$\Rightarrow G(s)H(s)=-1+j0$

ThephaseangleofG(s)H(s)is

$\angle G(s)H(s)=\tan^{-1}0/(-1)=(2n+1)\pi$

Theangleconditionisthepointatwhichtheangleoftheopenlooptransferfunctionisanoddmultiple of180^0.

MagnitudeofG(s)H(s)is-

$|G(s)H(s)|=\sqrt{(-1)^2+0^2}=1$

Themagnitudeconditionisthatthepoint(whichsatisfiedtheanglecondition)atwhichthemagnitudeoft heopen looptransferfunction is one.

RulesforConstruction ofRootLocus

Followtheserulesforconstructingarootlocus.

Rule1

−Locatetheopenlooppolesandzerosinthe‘s’plane.Rule2−Find

thenumberofrootlocusbranches.

We know that the root locus branches startat the open loop poles and endeither at openloop zeros or at ∞. So, the number of root locus branches N is equal to the number of finiteopenlooppolesP orthenumberoffinite openloop zerosZ,whicheverisgreater.

Mathematically, we can write the number of root locus branches N

as$N=P$if$P\geq Z$

$N=Z$ if$P<Z$

Rule3− Identifyanddrawthe realaxisrootlocusbranches.

A point or segment on the real axis lies on the root locus if the sum of open loop poles andopenloopzerostotheright ofthis pointorsegmentis odd.

Rule4−Findthecentroidand theangleofasymptotes.

Asymptotesgivethedirectionoftheserootlocusbranches.Numbe r ofAsymptotes=P-Z

The intersection point of asymptotes on the real axis is known as centroid.Wecancalculatethecentroid$\boldsymbol{\sigma_A}$byusingthisformula,

$\sigma_A = \frac{\sum \text{Real part of finite open loop poles} - \sum \text{Real part of finite open loop zeros}}{P-Z}$

Theformulafortheangleofasymptotesis $\Phi_A = \frac{(2q+1)180}{P-Z}$

Where,

$q=0,1,2,...,(P-Z-1)$

Rule5−Find the intersectionpointsofroot locusbrancheswithanimaginaryaxis.

Wecancalculatethepointatwhichtherootlocusbranchintersectstheimaginaryaxisandthevalueof Katthatpointbyusing theRoutharraymethod

Rule6−FindBreak-awayandBreak-inpoints.

- If there exists a real axis root locus branch between two open loop poles, then therewillbeabreak-away point in betweenthesetwoopenlooppoles.
- Ifthereexistsarealaxisrootlocusbranchbetweentwoopenloopzeros,thentherewillbeabreak-inpointinbetweenthesetwoopenloopzeros.

Note−Break-awayandbreak-

inpointsexistonlyontherealaxisrootlocusbranches.Followthesesteps tofind break-

awayandbreak-inpoints.

- WriteK intermsofsfromthecharacteristicequation$1+G(s)H(s)=0$

- Differentiate K with respect to s andmakeitequal tozero. Substitute these valuesofsintheaboveequation.
- The values of s for which the K value is positive are the break

points.Rule7−Find theangleofdepartureandtheangleofarrival.

TheAngleofdepartureandtheangleofarrivalcan becalculatedatcomplexconjugateopenlooppolesandcomplexconjugateopenloopzerosrespectively.

Theformulafortheangleofdeparture$\phi_d$is

Φ_d=180–sumoftheanglesofvectorsdrawntothispoletootherpoles+sumoftheanglesofvectors drawntothis poletozeros

Theformulafortheangleofarrival$\phi_a$is

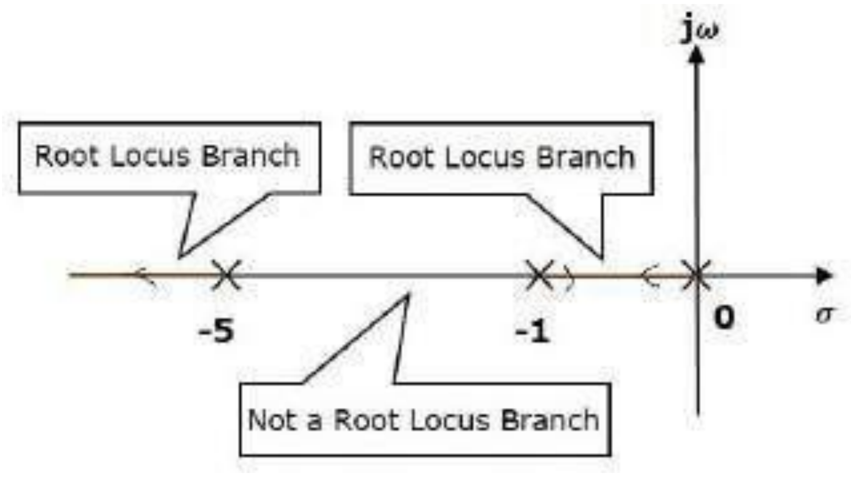
Φ_a =180–sumoftheanglesofvectorsdrawntothiszerotootherzeros+sumoftheanglesofvectorsdrawntothis zerotopoles

Example

Letusnowdrawthe rootlocusofthecontrolsystemhavingopenlooptransferfunction$G(s)H(s)=\frac{k}{s(s+1)(s+5)}$

Step 1 − The given open loop transfer function has three poles at s=0, s=−1 and s=−5. Itdoesn't have any zero. Therefore, the number of root locus branches is equal to the numberofpoles oftheopenloop transferfunction.

N=P=3



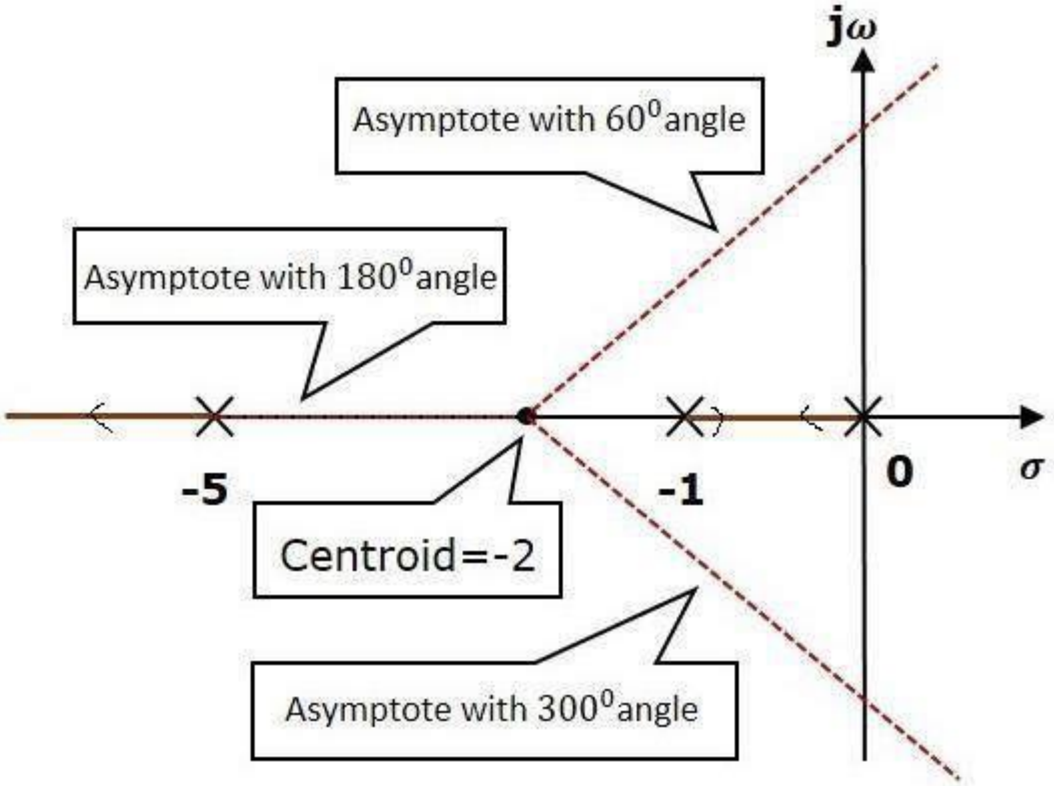
Thethreepolesarelocatedareshownintheabovefigure.The line segmentbetween s=−1 and s=0 is one branch of root locus on real axis. And the other branch of therootlocusontherealaxisisthelinesegment totheleftofs=−5i.ein between-5 and ∞.

Step 2 − We will get the values of the centroid and the angle of asymptotes by using thegivenformulae.

Centroidα $=\frac{0-1-5}{3-0}=-2$

TheangleofasymptotesΦ_A= $\frac{(2q+1)180}{P-Z} = \frac{(2q+1)180}{3-0}$ forq=0,1,2angleofasymptotes areθ=60°,180°and300°

Thecentroid andthreeasymptotesareshowninthefollowingfigure.



Step 3 − Since two asymptotes have the angles of 60^o and 300^o, two root locus branchesintersect the imaginary axis. By using the Routh array method and special case(ii), theintersectsofrootlocus branchestothe imaginaryaxiscan befoundoutas below

ThecharacteristicsequationofthegivenTFis1+G(s)H(s)=0

$$\text{Or } 1+\frac{k}{s(s+1)(s+5)}=0$$

$$\text{Or} s^3+6s^2+5s+K=0$$

Routharray

s3	1	5
s2	6	k
s1	$\frac{30-k}{6}$	0
s0	k	

ForsystemstabilitythecoefficientofRouth'sarrayhavingpositiveandnonzerovaluehence:
K>0
$\frac{30-k}{6}>0$ork<30
TherangeofKforwhichthesystembecamestableis0<k<30Atk =30,
thesystem auxiliary equation is
$6s^2+30=0$
Or $s=\pm j\sqrt{5}$

Hencetherootlocus intersecttheimaginaryaxisat$\pm j\sqrt{5}$

Step 4 − There will be one break-away point on the real axis root locus branch between thepoles s=−1 and s=0. By following the procedure given for the calculation of break-awaypoint,

Thecharacteristicsequation $s^3+6s^2+5s+K=0$

OrK=$-(s^3+6s^2+5s)$

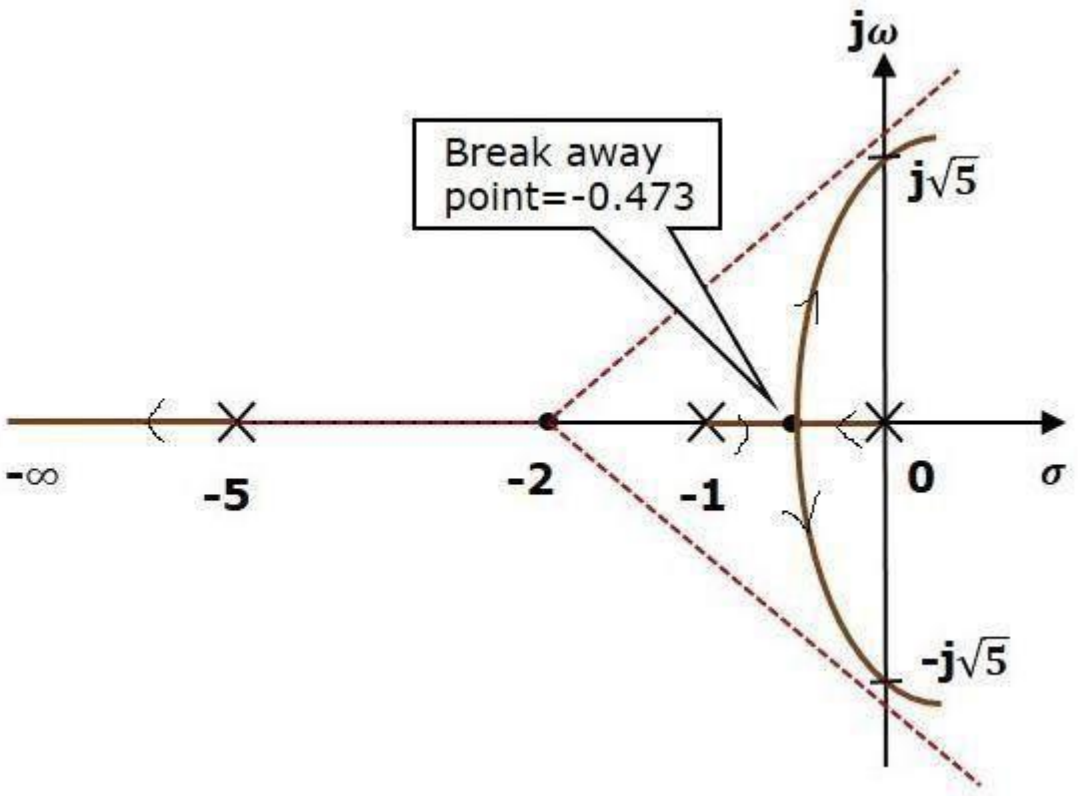
$\frac{dk}{ds}=0$

Or3 $s^2+12s+5=0$

Therootsofs=-0.473,-3.52

Sincebreakawaypointmustliebetween0and-1,itisclearthat s=−0.473is actualbreakawaypoint.

Therootlocusdiagramforthegivencontrolsystemis showninthefollowingfigure.



<u>Example</u>:- Afeedbackcontrolsystemhasopen-looptransferfunction

$$G(s)H(s) = \frac{K_a}{s(s+3)(S^2+2s+2)}$$

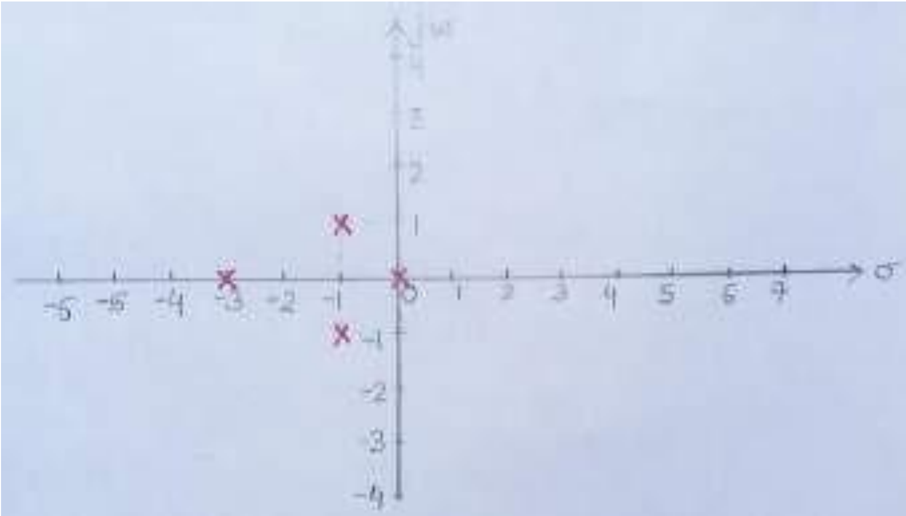
Draw root locus as K is varied from 0 to ∞

.Solution:

<u>Step-1</u>:-FindOLpoles andOLzeros fromtheOLTF

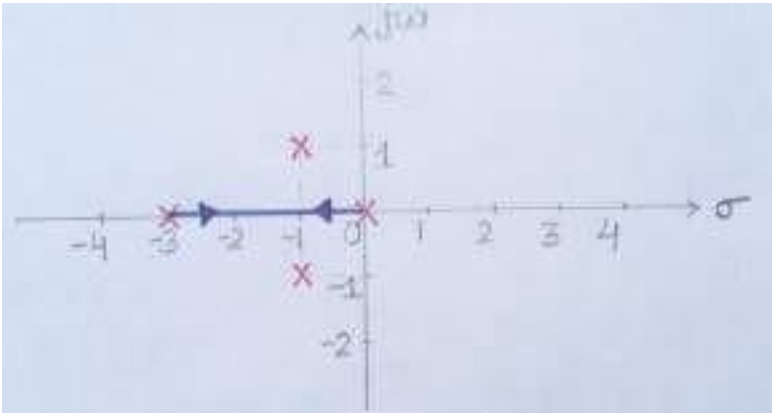
OL poles are S=0,-3, (-1+j1) and (-1-j1)Therearenofinite OLzeros.

MarkOLpolewithcrossand OLzerowithcircleinS-planeasshown.



Step-2Findthepartsoftherealaxisatwhichrootlocuslies.

A point on real axis lies on root locus if the number of OL poles+OL zeros on the real axis totherightofthepointisodd.HencetheRoot locusexistbetweens=0ands=-3intherealaxis.



Step-3 Number of root locus branches N = P= 4Step-4Findnumberofasymptotes:

Number of asymptotes = P - Z = 4 (where P,Z = nos of open loop pole and zero)Step-5Calculation forcetroid

$$\sigma_A = \frac{\sum \text{Realpartoffiniteopenlooppoles} - \sum \text{Realpartoffiniteopenloopzeros}}{P-Z}$$

$$= \frac{(0-3-1-1)-(0)}{4} = -1.25$$

Step-6Calculationforasymptoticangle:

$$\Phi_A = \frac{(2q+1)180}{P-Z}$$ For

q=0;Forq= 1; $\Phi_A = \frac{180\,(0+1)}{4} = 45^0$

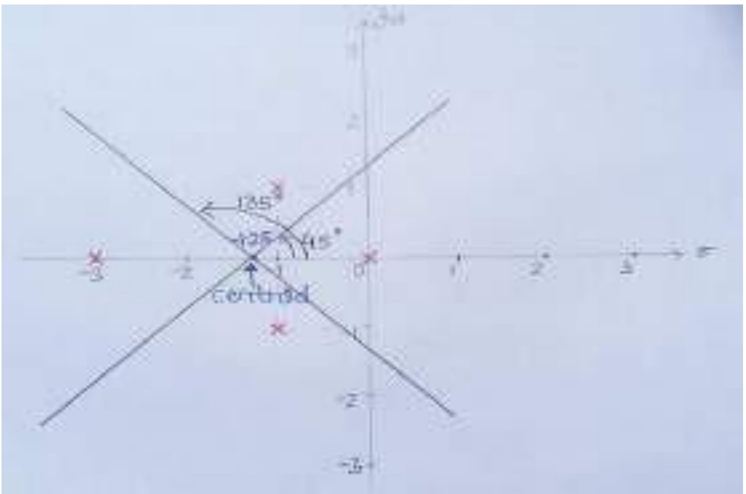
$\Phi_A = \frac{180\,(2+1)}{4} = 135^0$

Forq=2; $\Phi_A = 225^0$

$\Phi_A = 315^0$

For $q=3$;

So,fromsteps2,3and4,fourasymptotescuttherealaxisat-1.25andmakeangles45^0,135^0,225^0and315^0,as shownbelow.



Step-7:

Find the*breakaway points*(points at which two or more root locus branches meet)

Breakawaypointsarethe solutions of $(dK_a/ds)=0$

Thecharacteristic equationwillbe

$$S(S+3)(S^2+2S+2)+K_a=0$$

Fromthecharacteristicequation,

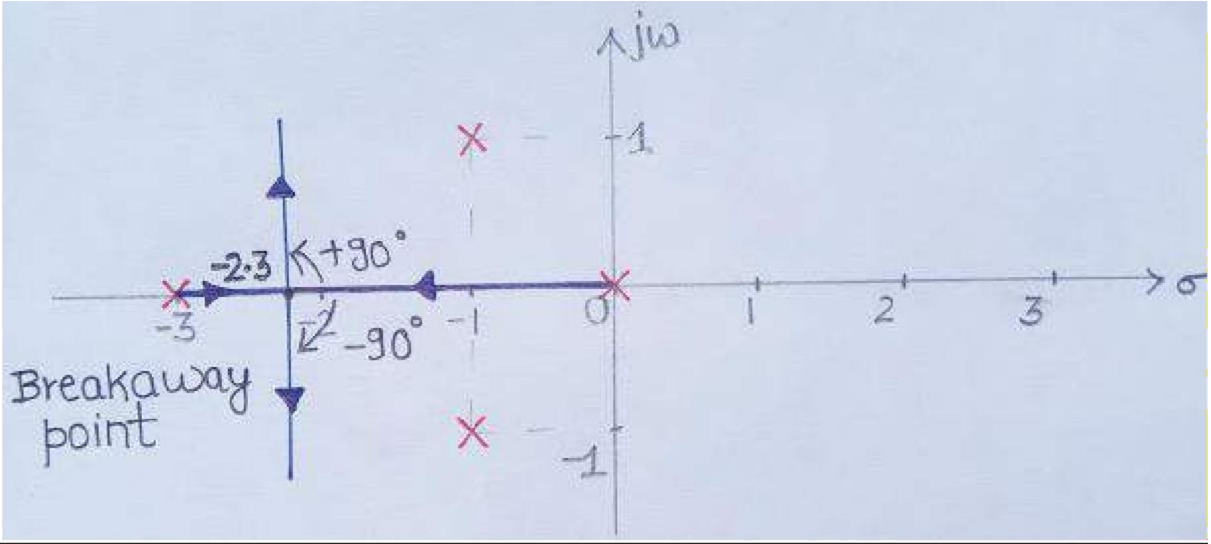
$K_a=-S(S+3)(S^2+2S+2)=-(S^4+5S^3+8S^2+6S)$

$$\therefore \frac{dK_a}{ds} = -4(S^3+3.75S^2+4S+1.5) = 0$$

Weget,$S=-2.3$,$(-0.725\pm j0.365)$

Not all values obtained as solutions of $(dK_a/ds)=0$ need to be necessarily the breakawaypoints. Out of the obtained s values only those values of S which satisfy angle condition aretheactualbreakawaypoints.

Oncheckinganglecondition wefindthat$(-0.725\pm j0.365)$donotsatisfyit.Therefore,only$S=-2.3$ is the only breakaway point. So, the real axis from 0 to -3 contains root locus whichbreakdownat-2.3 as shown.



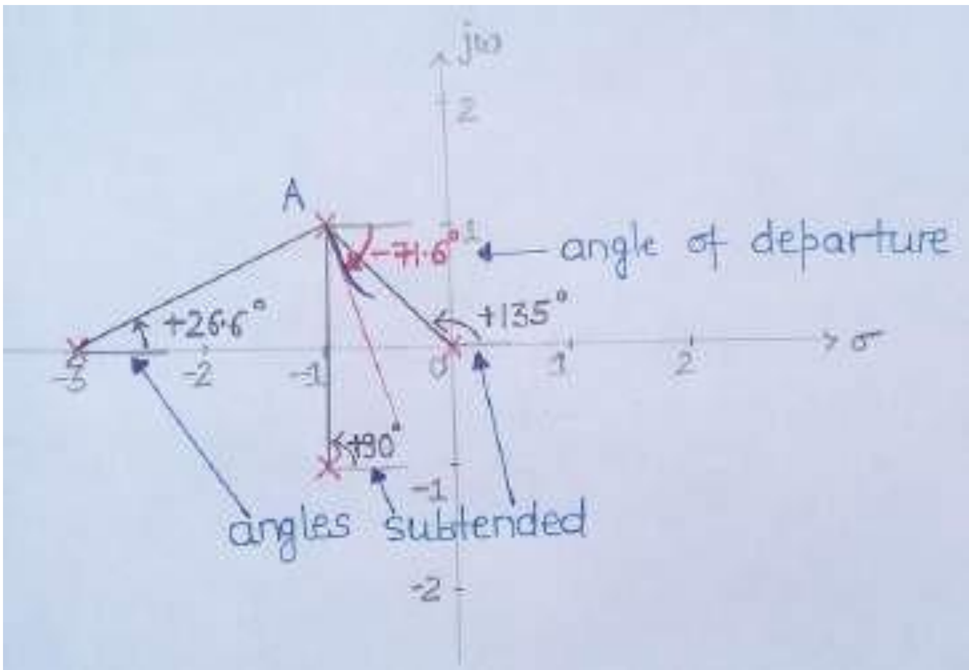
Step-8 :- Find *angles of departure* as there is a presence of pole in complex plane (anglewhicharootlocusbranchstartingfromanopenlooppole,makeswithalineparalleltotheasympt oticline.

Theformulafortheangleofdeparture$\phi_d$is

Φ_d=180–sumoftheanglesofvectorsdrawntothispoletootherpoles+sumoftheanglesofvectors drawntothis poletozeros

$$\text{Or}\Phi_d=180-(90^0+135^0+26.6^0)=-71.6^0$$

So,rootlocusbranchstartsfrom(-1+j1)atanangle-71.6^0.Sincerootlocusisalwaysmirrorimageaboutrealaxis,therefore,root locusstarts from(-1-j1)at+71.6^0.



Step-9:-Findthepointsatwhichrootlocusbranches intersectjwaxis.Thecharacteristic equationwillbe

$$S(S+3)(S^2+2S+2)+K_a=0$$

Or$S^4+5S^3+8S^2+6S+K_a=0$, Makerouthsarray;

S^4	1	8	K_a
S^3	5	6	
S^2	$\frac{(5\times8)-(6\times1)}{5}=6.8$	K_a	
S^1	$\frac{(6.8\times6)-(Ka\times 5)}{6.8}$		
S^0	K_a		

ForthesystemtobestableallthecoefficientofthefirstcolumnoftheRouth'sarrayhavingpositiveand nonzerovalue.Henceforsystemstability

$$K_a>0$$

$$\frac{(6.8\times6)-(Ka\times5)}{6.8}>0$$

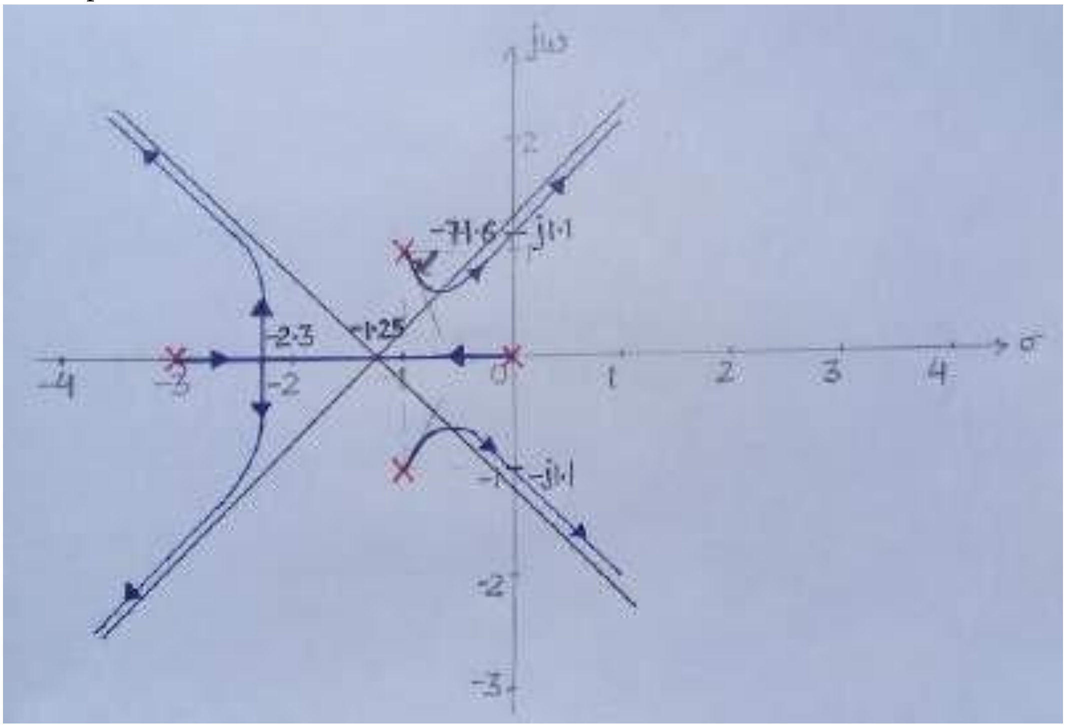
$$\text{Or}\,0<K_a<8.16$$

For K_a=8.16the Auxiliaryequation is$6.8s^2+8.16=0$

$$\text{Or}\,s^2=-1.2$$

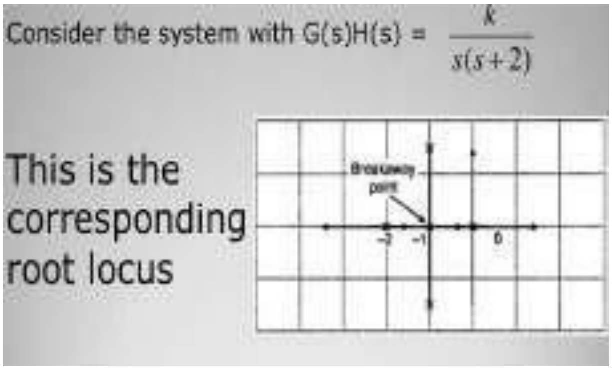
$$\text{Or}\,s=\pm j1.1$$

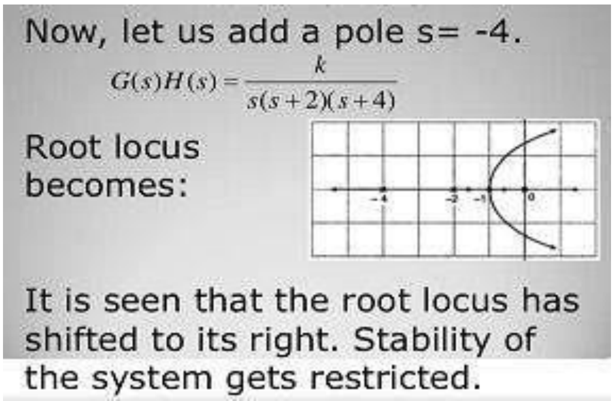
The points of intersection comes out to be +j1.1 and –j1.1Thecomplete rootlocusis shownbelow.

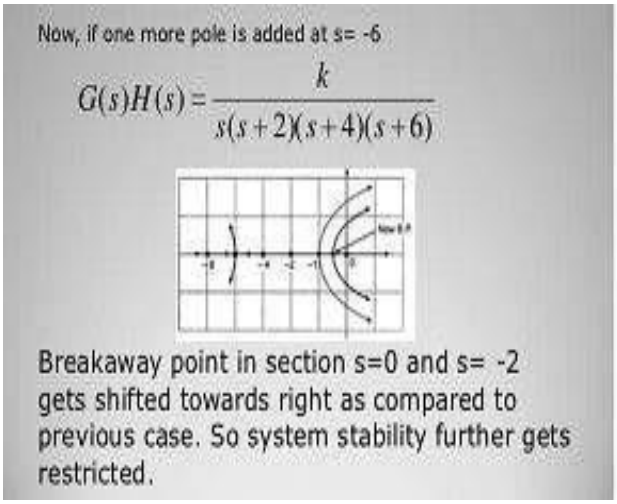


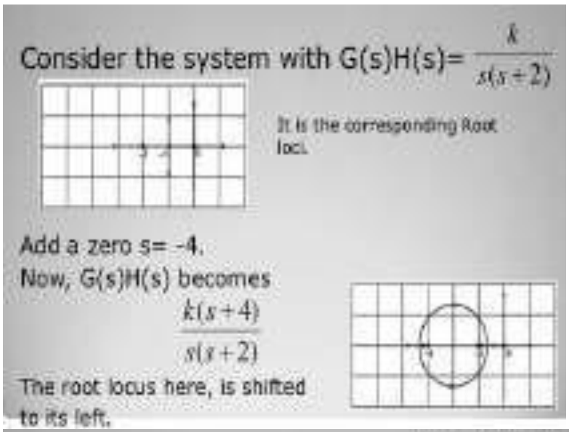
EffectsofAdding OpenLoopPoles andZerosonRootLocus

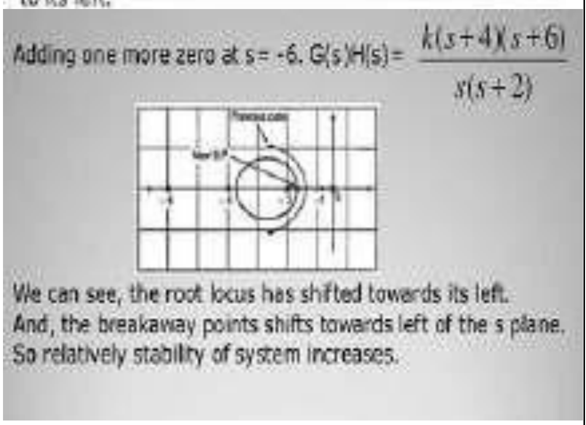
Effectofadditionofopen looppole	Effectofadditionofopen loopzero











➢The root locus changes its nature and shifts towards imaginary axis. ➢The system becomes oscillatory. ➢Gain Margin enhances relatively, thus stability decreases. ➢Range of k reduces. ➢Settling time increases.	➢The root locus changes its nature and shifts to left away from imaginary axis. ➢Relative stability of system increases. ➢System becomes less oscillatory. ➢Gain margin increases and so does the range of K. ➢Settling time decreases.

FrequencyResponseAnalysis

Introduction:

The response of a system can be partitioned into both the transient response and the steady stateresponse. We can find the transient response by using Fourier integrals. The steady state responseof a system for an input sinusoidal signal is known as the frequencyresponse. In this chapter, wewillfocus onlyon the steadystateresponse.

If a sinusoidal signal is applied as an input to a Linear Time-Invariant (LTI) system, then itproducesthesteadystateoutput,whichisalsoasinusoidalsignal.Theinputandoutputsinusoidalsignals havethesame frequency, but differentamplitudes andphaseangles.

Lettheinputand outputsignalbe−

$$r(t)=A\sin(\omega t) \tag{1}$$

$$c(t)=B\sin(\omega t+\phi) \tag{2}$$

Theopenloop transferfunctionwillbe−

$$G(s)=G(j\omega)$$

WecanrepresentG(jω)intermsofmagnitudeandphaseasshownbelow.

$$G(j\omega)=|G(j\omega)|\angle G(j\omega)$$

Theoutputsignal is

$$c(t)=A|G(j\omega)|\sin(\omega t+\angle G(j\omega)) \tag{3}$$

- Theamplitudeoftheoutputsinusoidalsignalisobtainedbymultiplyingthe amplitudeoftheinput sinusoidal signal and the magnitudeof G(jω)at ω.
- The phase of theoutputsinusoidalsignalisobtainedby addingthe phase ofthe inputsinusoidalsignal and the phaseof G(jω)at ω

Where,

- Ais theamplitudeoftheinputsinusoidalsignal.
- Ωisangularfrequencyoftheinputsinusoidalsignal.Wec

an write,angularfrequencyωasshown below.

$$\omega=2\pi f$$

Here, f is thefrequencyoftheinput sinusoidalsignal. Similarly, you can followthesameprocedurefor closed loop control system.

Correlationbetweentimeandfrequencyresponse:

The frequency domain specifications are resonant peak, resonant frequency and

bandwidth.Considerthetransfer functionofthe secondorderclosed loopcontrol systemas,

$$T(s)=C(s)/R(s)=\omega_n^2/(s^2+2\zeta\omega_n s+\omega_n^2)$$

Substitute, $s=j\omega$ in the above equation.

$$T(j\omega)=\omega_n^2/(j\omega)^2+2\zeta\omega_n(j\omega)+\omega_n^2$$

$$\Rightarrow T(j\omega)=\frac{\omega_n^2}{-\omega^2+2j\zeta\omega\omega_n+\omega_n^2}=\frac{\omega_n^2}{\omega_n^2\left(-\frac{\omega^2}{\omega_n^2}+2j\zeta\left(\frac{\omega}{\omega_n}\right)+1\right)}$$

$$\Rightarrow T(j\omega)=\frac{1}{\left(1-\frac{\omega^2}{\omega_n^2}\right)+j2\zeta\left(\frac{\omega}{\omega_n}\right)} \qquad (4)$$

Let, $\frac{\omega}{\omega_n}=u$ Substitute this value in the above equation.

$$T(j\omega)=\frac{1}{(1-u^2)+j2\zeta u} \qquad (5)$$

Magnitude of $T(j\omega)$ is -

$$M=|T(j\omega)|=\frac{1}{\sqrt{(1-u^2)^2+(2\zeta u)^2}} \qquad (6)$$

Phase of $T(j\omega)$ is -

$$\angle T(j\omega)=-\tan^{-1}\frac{2\zeta u}{(1-u^2)} \qquad (7)$$

The steady-state output of the system for a sinusoidal input of unit magnitude and variable frequency ω is given by

$$C(t)=\frac{1}{\sqrt{(1-u^2)^2+(2\zeta u)^2}}\sin\left(\omega t-\tan^{-1}\frac{2\zeta u}{(1-u^2)}\right) \qquad (8)$$

Resonant Frequency:

It is the frequency at which the magnitude of the frequency response has peak value for the first time. It is denoted by ω_r. At $\omega=\omega_r$ the first derivate of the magnitude of $T(j\omega)$ is zero.

Differentiate M with respect to u.

$$\frac{dM}{du}\Big|_{u=u_r}=-\frac{1}{2}\frac{-4(1-u_r^2)u_r+8\zeta^2u_r}{\left[(1-u_r^2)^2+(2\zeta u_r)^2\right]^{3/2}}=0$$

$$\Rightarrow 4u_r^3-4u_r+8\zeta^2u_r=0$$

$$\Rightarrow u_r=\sqrt{1-2\zeta^2} \qquad (9)$$

$$\text{i.e., } \omega_r=\omega_n\sqrt{1-2\zeta^2} \qquad (10)$$

Resonant Peak:

It is the peak (maximum) value of the magnitude of $T(j\omega)$. It is denoted by M_r. At $u=u_r$, the Magnitude of $T(j\omega)$ is -

$$M = |T(j\omega)| = \frac{1}{\sqrt{(1-u^2)^2 + (2\zeta u)^2}}$$

Substitute, $u_r = \sqrt{1-2\zeta^2}$ in the above equation, we get

$$M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \qquad (11)$$

The phase angle of $T(j\omega)$ at resonant frequency u_r obtained from equation 7 is

$$\Phi_r = -\tan^{-1}[\sqrt{1-2\zeta^2}/\zeta] \qquad (12)$$

Resonant peak in frequency response corresponds to the peak overshoot in the time domain transient response for certain values of damping ratio ζ. So, the resonant peak and peak overshoot are correlated to each other.

Bandwidth:

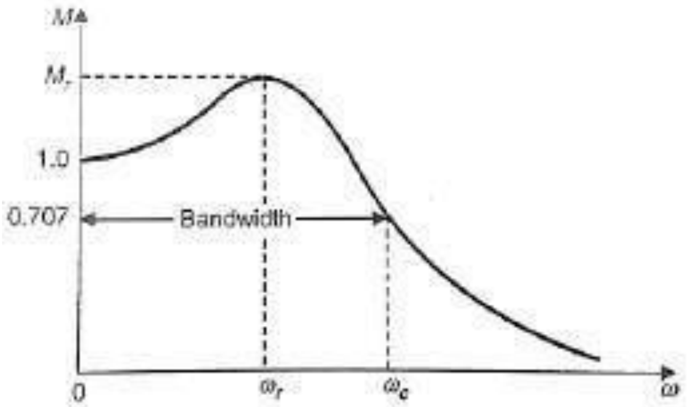
It is the range of frequencies over which, the magnitude of $T(j\omega)$ drops to 70.7% (0.707) from its zero frequency value.

At $\omega=0$, the value of u will be zero. Substitute, $u=0$ in M, frm equation 6

$$M = \frac{1}{\sqrt{(1-0^2)^2 + (2\zeta 0)^2}} = 1$$

Therefore, the magnitude of $T(j\omega)$ is one at $\omega=0$.

At 3-dB frequency, the magnitude of $T(j\omega)$ will be 70.7% of magnitude of $T(j\omega)$ at $\omega=0$. i.e., at $\omega=\omega_b$, $M = 0.707(1) = 1/\sqrt{2}$



Typical magnification curve of a feedback control system.

From Equation 6: $\quad M = \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{(1-u_b^2)^2 + (2\zeta u_b)^2}}$

$$\Rightarrow 2 = (1-u_b^2)^2 + (2\zeta)^2 u_b^2$$

Let, $u_b^2 = x$

$$\Rightarrow 2 = (1-x)^2 + (2\zeta)^2 x \Rightarrow x^2 + (4\zeta^2 - 2)x - 1 = 0$$

$$\Rightarrow x = \frac{-(4\zeta^2-2) \pm \sqrt{(4\zeta^2-2)^2-4}}{2}$$

Consideronlythepositive valueofx.

$$x = \frac{-(4\zeta^2-2) + \sqrt{(4\zeta^2-2)^2-4}}{2}$$

or

$$X = 1-2\zeta^2\sqrt{2-4\zeta^2+4\zeta^4}$$

Substitute, $x = u^2 = \omega_b^2/\omega_n^2$

$$\omega_b^2/\omega_n^2 = 1-2\zeta^2\sqrt{2-4\zeta^2+4\zeta^4}$$

$$\Rightarrow \omega_b = \omega_n\sqrt{1-2\zeta^2\sqrt{2-4\zeta^2+4\zeta^4}} \tag{13}$$

Bandwidth ω_b inthefrequencyresponseisinverselyproportionaltotherisetime t_r inthetimedomaintransient response.

<u>BodePlots</u>

Sinusoidal transfer function is graphically represented by Bode plot for determining the stabilityofthecontrol system. Bodeplot is a logarithmicplotand consists of two plots.

- Aplotofthelogarithmic(base10)ofmagnitude(indecibel)Vsfrequencyin logarithmicscalei.e. logω.
- A plotofPhaseplot(ϕ)Vs frequencyin logarithmic scale i.e.logω.

Inboththeplots,x-axisrepresentsangularfrequency(logarithmicscale).Whereas,y-axisrepresents the magnitude (linear scale) of open loop transfer function in the magnitude plot andthephaseangle(linear scale)oftheopen loop transfer functionin thephaseplot.

Themagnitudeoftheopenlooptransfer functionindB (decibel)is-

$$M=20\log|G(j\omega)H(j\omega)| \tag{1}$$

Thephaseangleoftheopenloop transferfunction indegrees is-

$$\phi=\angle G(j\omega)H(j\omega) \tag{2}$$

Note−thebaseoflogarithm is 10.

BasicofBodePlots

Letthegeneralisedexpressionforopen-looptransferfunctionofasystembegivenby:

$$G(s)\,H(s)=\frac{K\left[(1+sT_1)(1+sT_2)\ldots\ldots\right]\omega_n^2}{s^n\left[(1+sT_a)(1+sT_b)\ldots\ldots\right]\left[s^2+2\xi\omega_n s+\omega_n^2\right]} \tag{3}$$

where K, T_1, T_2, ..., T_a, T_b, ..., ξ, ω_n are ali real coefficients

Puts=jωin equation3,weget

$$G(j\omega)H(j\omega) = \frac{K[(1+j\omega T_1)(1+j\omega T_2)\ldots]}{(j\omega)^n[(1+j\omega T_r)(1+j\omega T_b)\ldots]\left[1+j2\xi\left(\frac{\omega}{\omega_n}\right)+\left(\frac{j\omega}{\omega_n}\right)^2\right]} \quad (4)$$

$$= \frac{K[(1+j\omega T_1)(1+j\omega T_2)\ldots]}{(j\omega)^n[(1+j\omega T_a)(1+j\omega T_b)\ldots][1+j2\xi u-u^2]} \quad (5)$$

Whereu=ω/ω_n

Fromequation5,themagnitudeofG(jω)H(jω)indecibels isgivenby

From equation (11.9), the magnitude of G (jω) H (jω) in decibels is given by

$$20\log_{10}|G(j\omega)H(j\omega)| = 20\log_{10}K + (20\log_{10}|1+j\omega T_1| + 20\log_{10}|1+j\omega T_2|\ldots)$$
$$-[20n\log\omega + 20\log|1+j\omega T_a| + 20\log|1+j\omega T_b| + \ldots] - 20\log_{10}|(1-u^2)+j2\xi u| \quad (6)$$

$$\text{or}\quad 20\log_{10}|G(j\omega)H(j\omega)| = [20\log K + 20\log\sqrt{1+\omega^2T_1^2} + 20\log\sqrt{1+\omega^2T_2^2} + \ldots]$$
$$-20N\log\omega - 20\log\sqrt{1+\omega^2T_a^2} - 20\log\sqrt{1+\omega^2T_b^2}\ \ldots$$
$$\ldots - 20\log\sqrt{(1-u^2)^2+4\xi^2u^2} \quad (7)$$

and phase angle of G (jω) H (jω) is given by

$$\angle G(j\omega)H(j\omega) = \angle K + \angle(1+j\omega T_1) + \angle(1+j\omega T_2) + \ldots$$
$$-\angle(j\omega)^n - \angle(1+j\omega T_a) - \angle(1+j\omega T_b)\ldots$$
$$-\angle(1-u^2+j2\xi u)$$
$$\angle G(j\omega)H(j\omega) = 0^\circ + \tan^{-1}\omega T_1 + \tan^{-1}\omega T_2 + \ldots \quad (8)$$
$$-(90\times n) - \tan^{-1}\omega T_a - \tan^{-1}\omega T_b\ldots$$
$$-\tan^{-1}\left(\frac{2\xi u}{1-u^2}\right) \quad (9)$$

Generally there are the following seven simple types of factors in G (jω) H (jω) :

(i) Constant K
(ii) Zeros at origin $(j\omega)^{+n}$
(iii) Poles at the origin $(j\omega)^{-n}$
(iv) Simple zero on real axis $(1 + j\omega T)$
(v) Simple pole on real axis $\frac{1}{(1+j\omega T)}$
(vi) Complex conjugate pole $\frac{1}{(1+j2\xi u-u^2)}$
(vii) Complex conjugate zero $(1 + j2\xi u - u^2)$

Procedureforplotting Bodeplot:

Step1:Rewritetheopenlooptransferfunctioninthetimeconstant formasgiveninequation4.Step2:Identifythecornerfrequenciesassociatedwitheachfactorofthetransf erfunction.

Step3:Afterknowingthecornerfrequencies,drawtheasymptoticmagnitudeplot.Thisplotconsistsof straightlinesegments withlineslopechangingateachcornerfrequencyasfollows.

(i)+20db /decade fora zero and+20ndb/decadeforazero ofmultiplicityn.

(ii)-20db/decadefor apole and-20n db/decadefor apole of multiplicityn.

(iii) + 40db/decade for a complex conjugate zero and + 40n db/decade for a complexConjugatezero ofmultiplicityn.

(iv)-40db/decade for a complex conjugate pole and - 40n db/decade for a complexConjugatepole ofmultiplicityn.

Step3: InitialslopeofBodeplotarecalculatedasfollows.

(i) For type zero system draw a line up to first (lowest) corner frequency having 0db/decadeslope.

(ii) Fortypeonesystemdrawalinehavingslopeof -20db/decadeupto ω=K.Markfirst(lowest)cornerfrequency.

(ii)Fortype twosystem drawalinehavingslopeof-40db/decadeup to ω=√K andsoon.Markfirst(lowest) cornerfrequency.

Step4:Drawalineuptosecondcornerfrequencybyaddingtheslopeofnextpoleorzerotothepreviousslope and so on.

Step5:Calculatephaseanglefordifferentvaluesofωfromtheequation 9and joinall points.

Note−Thecornerfrequency(ω=1/K)isthefrequencyatwhichthereisachangeintheslopeofthemagnitude plot.

Example1: Drawthe bodeplotforunityfeedbackcontrolsystemhaving$G(s)=\frac{1000}{(s+100)}$.

Solution:

Step1:Open-looptransferfunctionintimeconstantformisgivenby

$$G(s)H(s)=\frac{1000}{(s+100)}$$

$$=\frac{1000}{100\frac{(s+100)}{100}}=\frac{10}{(1+0.01s)} \text{ (Timeconstantform)}$$

Puts=jω $$G(j\omega)H(j\omega)=\frac{10}{(1+j0.01\omega)}$$

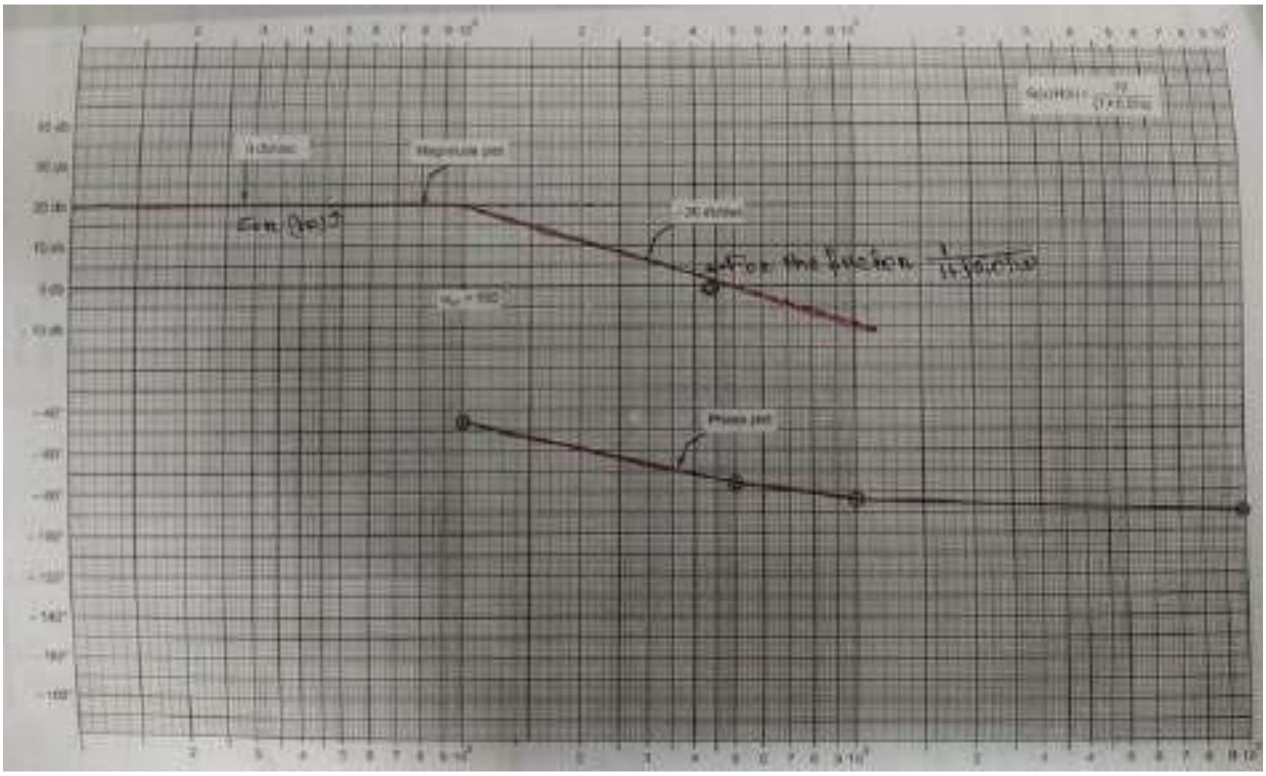
Step2:Cornerfrequencyω=1/0.001=100

Step3:Thereisonepoleonthe realaxishencemagnitudeplotisastraightlinehavingslopeof-20db/decade.

Step4:Asthesystem istypezerosystemso magnitudeplot is astraightlineparallelto odbaxis andhavingmagnitude $20\log_{10}K=20\log_{10}10=20$db.

Step5:phase angle$\phi = -\tan^{-1}0.01\omega$.The tableshows value ofϕ when ωvaries from 0 to∞.

S. No.	ω	φ
1.	0	0°
2.	10	– 5.7°
3.	100	– 45°
4.	500	– 78.7°
5.	1000	– 84.29°
6.	10,000	– 90°



Example 2: Draw the bode plot for unity feedback control system havingG(s)=5(s+2)/s(s+10)

Puts=jω

$$G(j\omega)= \frac{5(j\omega+2)}{j\omega(j\omega+10)} = \frac{5\times 2\left(1+\frac{j\omega}{2}\right)}{10j\omega\left(1+\frac{j\omega}{10}\right)} = \frac{1+\frac{j\omega}{2}}{j\omega\left(1+\frac{j\omega}{10}\right)}$$

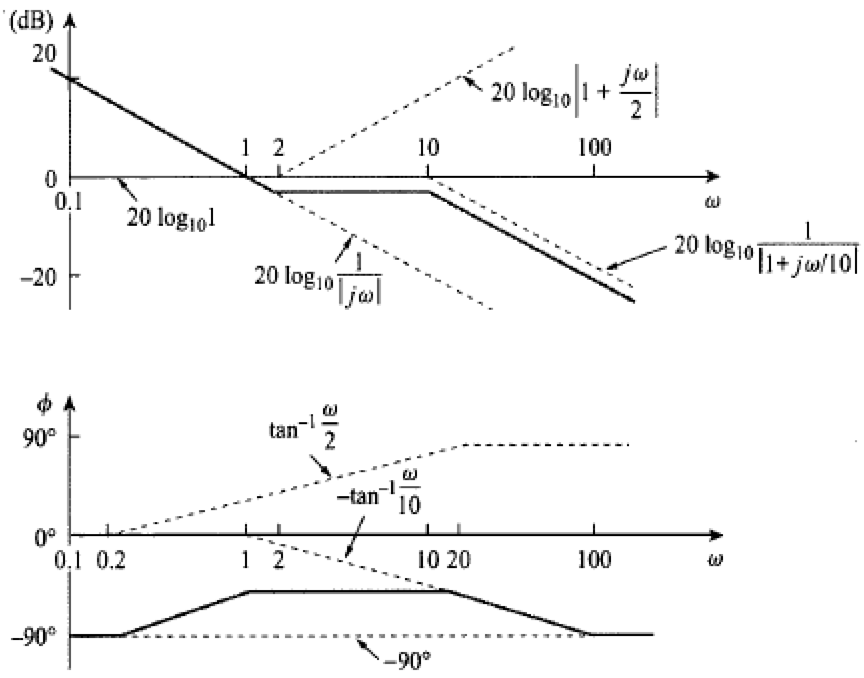
$$|G(j\omega)|\angle G(j\omega) = \frac{\left|1+\frac{j\omega}{2}\right|}{|j\omega|\left|1+\frac{j\omega}{10}\right|}\angle\left[90^\circ+\tan^{-1}\left(\frac{\omega}{2}\right)-\tan^{-1}\left(\frac{\omega}{10}\right)\right]$$

Magnitudeplot:

Sl. No	Factor	CornerFrequency	Slope	Asymptoticlogmagnitude
1	$\frac{1}{j\omega}$	None	-20 db/decade	Straightlineofslope-20db/decadeandintersecting the 0 dB axis at ω=K=1and extenduptofirst corner frequency2.
2	1+0.5$j\omega$	2	+20 db/decade	Draw anetslope(-20) +(+20)= 0db/decade fromcornerfrequency2tothenextcornerfrequency10.
3	$\frac{1}{1+0.1\,j\omega}$	10	-20 db/decade	Drawthenetslopeof0+(-20)=-20db/decadefromcornerfrequency10to ∞.

Note:Arrangethetableinincreasingorderofcornerfrequency.

Fordifferentvalueofωcalculatephaseangle∠G(jω) andjoinall thepointsbyfreehand.



ComputationofGainMarginandPhaseMargin

FromtheBodeplots,wecansaywhetherthecontrolsystemisstable,marginallystableorunstablebased on the values of theseparameters.

- Gaincrossoverfrequencyandphasecrossover frequency
- Gainmarginandphasemargin

PhaseCrossoverFrequency

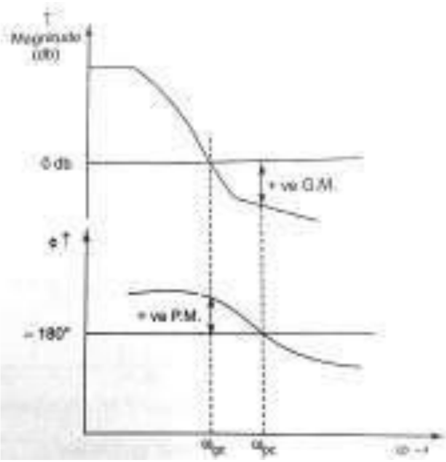
Thefrequency atwhichthephaseplot ishaving thephaseof-180^0 isknownasphasecrossover **frequency**. Itisdenotedbyω_{pc}.Theunitofphasecrossover frequencyis **rad/sec**.

GainCrossoverFrequency

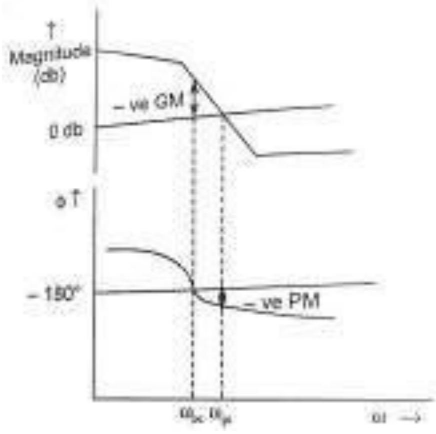
Thefrequencyatwhichthemagnitudeplotishavingthemagnitudeofzerod Bisknownasgaincrossoverfrequency.Itisdenotedbyω_{gc}.Theunitof gaincrossoverfrequencyisrad/sec.

Thestabilityofthecontrolsystembasedontherelationbetweenthephasecrossoverfrequencyandthegain cross over frequencyis listed below.

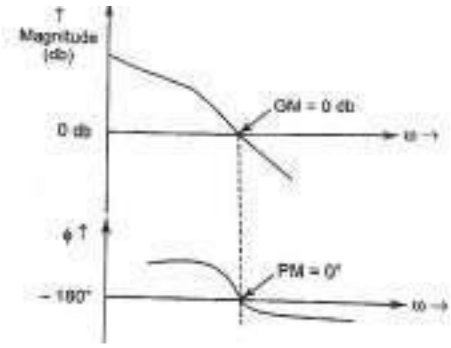
- Ifthephasecrossoverfrequencyω_{pc}isgreaterthanthegaincrossoverfrequencyω_{gc},thenthecontrol system isstable.
- Ifthephasecrossoverfrequencyω_{pc}isequaltothegaincrossoverfrequency ω_{gc},thenthecontrol system ismarginally stable.
- Ifthephasecrossoverfrequencyω_{pc}islessthanthegaincrossoverfrequency ω_{gc},thenthecontrol system isunstable.



ω_{pc}>ω_{gc}, GM&PMare+ve
StableSystem



ω_{pc}<ω_{gc}, GM &PM are–ve
Un-stableSystem



ω_{pc}=ω_{gc}, GM=PM=0
marginallystablesystem

GainMargin

Gain margin GM is defined as the margin in gain allowable by which gain can be increased tillsystem reaches on the verge of instability. It is equal to negative of the magnitude in dB at phasecrossoverfrequency. Mathematically

$$GM=20\log_{10}\left(\frac{1}{|G(j\omega)|_{\omega=\omega_{pc}}}\right)=-\,20\log_{10}\ |G(j\omega)|_{\ \omega=\omega_{pc}}$$

Theunit ofgain margin(GM)isdB.

PhaseMargin

Phasemargincanbedefinedastheamountofadditionalphaselagwhichcanbeintroducedinthesystemtillthesystemreacheson thevergeofinstability.TheformulaforphasemarginPMis

$$PM=[\angle G(j\omega)|_{\omega=\omega_{gc}}]-(-180^{o})$$

$$=180^{o}+[\angle G(j\omega)|_{\omega=\omega_{gc}}]$$

Theunitofphasemargin isdegrees.

Thestabilityofthecontrolsystembasedontherelationbetweengainmarginandphasemarginislisted below.

- If both the gain margin GM and the phase margin PM are positive, then the control systemis stable.
- If boththe gain margin GMand the phase margin PM are equaltozero, then the controlsystemismarginally stable.
- If the gain margin GM and / or the phase margin PM are/is negative, then the controlsystemis unstable.

Example3:Aunityfeedbackcontrolsystemhas

$$G(s)=\frac{20}{s(1+0.1s)(1+0.01s)}$$

Drawthebodeplot.FindGaincrossoverfrequency,phasecrossoverfrequency,gainmarginandphase margin.

Solution:Puts=jωinopenlooptransferfunction

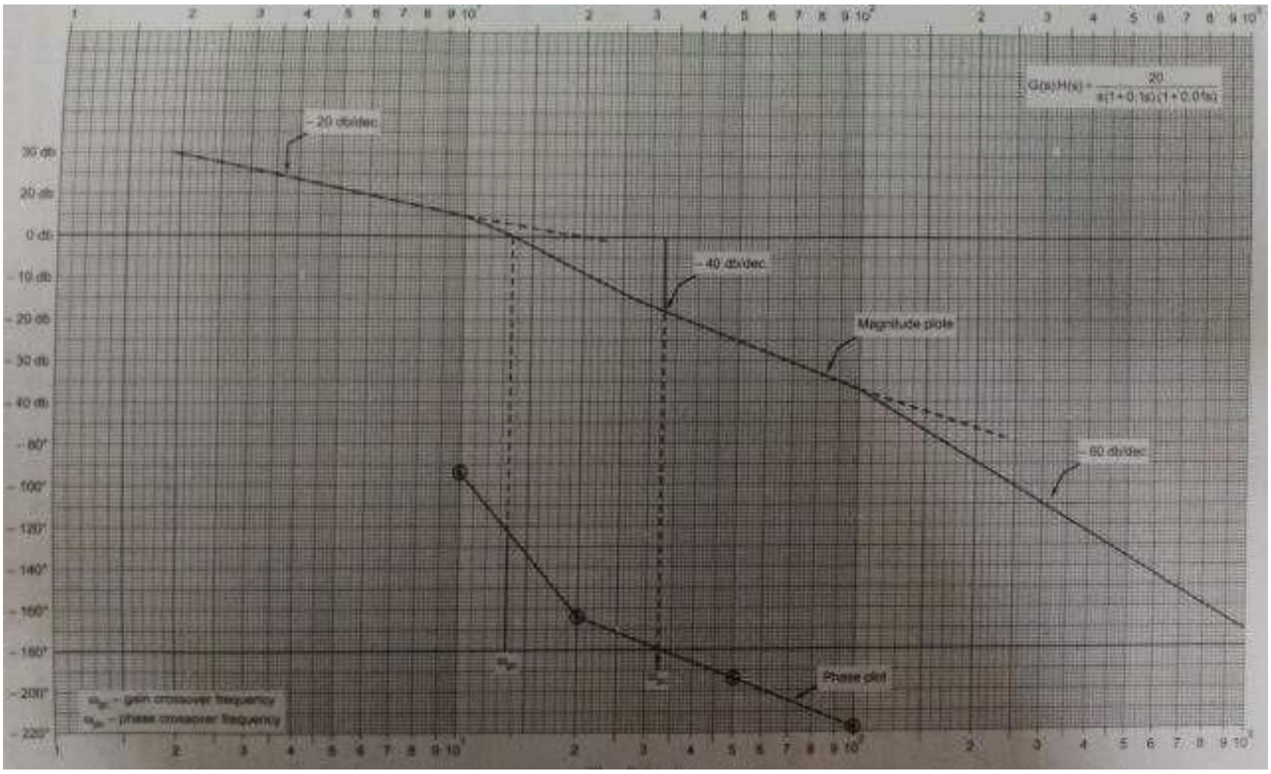
$$G(s)=\frac{20}{j\omega(1+0.1j\omega)(1+0.01j\omega)}$$

$$|G(j\omega)|\angle G(j\omega)=\frac{20}{-\omega^2\sqrt{1+(0.1\omega)^2}\sqrt{1+(0.01\omega)^2}}\angle -90° - tan^{-1}0.1\omega - tan^{-1}0.01\omega$$

Sl. No	Factor	CornerFrequency	Slope	Asymptoticlogmagnitude
1	$\frac{20}{j\omega}$	None	-20 db/decade	Straightlineofslope-20db/decadeandintersectingthe0dBaxisatω=K= 20andextendupto firstcorner frequency10.
2	$\frac{1}{1+0.1\,j\omega}$	10	20db/decade	Drawanetslope(-20)+(-20)=-40db/decade fromcornerfrequency10tothenextcornerfrequency100.
3	$\frac{1}{1+0.01\,j\omega}$	100	-20 db/decade	Drawthenetslopeof(-40)+(-20)=-60db/decadefromcornerfrequency100to∞.

Thetableshownbelow showsphaseangleforthe different valueofω.

S.No.	ω	ϕ
1.	0	– 90°
2.	10	– 95.71°
3.	20	– 164.74°
4.	50	– 195.25°
5.	100	– 219.3°



Fromtheplots

1. Gaincrossoverfrequencyω_{gc}=13.5
2. Phasecrossover frequencyω_{pc}=33
3. $\omega_{pc}>\omega_{gc}$, GM&PMare+ve,hencethesystembecomestable.
4. GainMargin = +15db
5. PhaseMargin=180^o-($+124^o$)=$+56^o$

Allpassandminimumphasesystem

If all the poles and zeros of any transfer function lie o the left half of s-plane, such type oftransfer function is known as*minimum phasetransfer function.*

Thetransferfunctionhavingapole-zeropatternwhichisantisymmetricabouttheimaginary axis i.e for every pole in the left half plane, there is a zero in the mirror image position.Thistypeof transfer function is known as*all passtransfer function.*

Acommonexampleofsuchtransferfunctionis

$$G(j\omega)=\frac{1-j\omega T}{1+j\omega T} \tag{1}$$

Polezeroconfigurationofequation1isshownbelow:

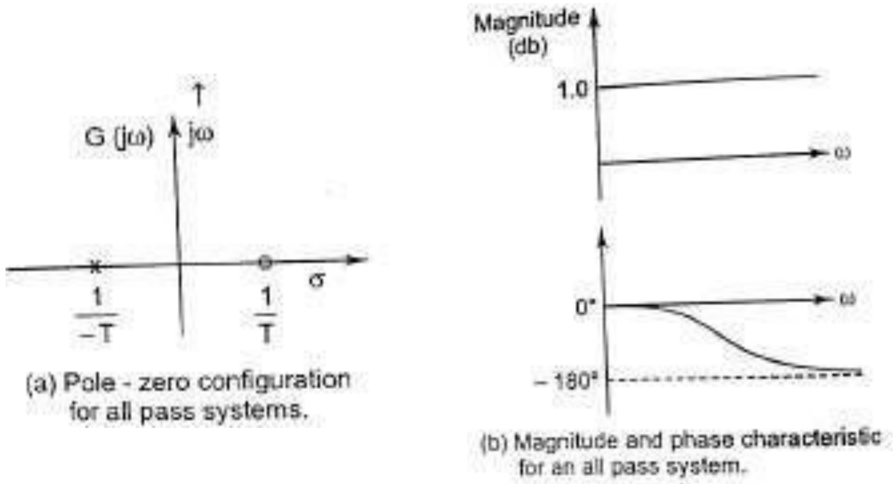


Figure1.All passsystem

All passtransferfunctionhasamagnitudeofunityatallfrequencyandaphaseangleof(- 2 $\tan^{-1}\omega T$)which varies from 0^o to -180^o as ωincreases from 0 to ∞. *The property of unitmagnitudeatallfrequenciesappliestoalltransferfunctionwithantisymmetricpole-zeropattern.*Physical systemswith this propertyarecalled all-pass system.

Now consider the casewhere the transfer function has poles in the left half s-plane andzero in both left and right half s-plane. Poles are not permitted to lie in the right half s-planebecausesuch asystem wouldbe unstable. Consider thefollowingtransferfunction

$$G_1(j\omega)= \frac{1-j\omega T}{(1+j\omega T_1)(1+j\omega T_2)} \qquad (2)$$

Whosepolezero pattern isshownin figure .Thistransferfunctionmayberewrittenas

$$G_1(j\omega)=\left[\frac{1+j\omega T}{(1+j\omega T_1)(1+j\omega T_2)}\right]\left[\frac{1-j\omega T}{1+j\omega T}\right]=G_2(j\omega)G(j\omega) \qquad (3)$$

Which is now become the product of two transfer function $G_2(j\omega)$ i.e minimum phase transferfunction shown in figure (2b) and $G(j\omega)$ i.e all pass transfer function shown in figure (2c). It isclear that $G_1(j\omega)$ and $G_2(j\omega)$ have identical curve of magnitude Vs frequency but their phase Vsfrequency curve are different as shown in figure(3). $G_2(j\omega)$ having a smaller range of phase anglethan $G_1(j\omega)$. A transfer function which has one or more zeros and no pole in the right half s-planeisknown as non-minimum phasetransfer function.

In general if the transfer function has any zeros in the right half s-plane, it is possible toextractthemonebyonebyassociatingthemwithall-passtransferfunctionasshowninfigure(2a).

A common example of a non-minimum phase element is transportation lag which hastransfer function

$G(j\omega)= e^{-j\omega T}=1\angle-\omega T$rad=

$1\angle-57.3\omega T$degreeOtherpossiblenon-minimum

transferfunction are:

1. wheremorethanonepossiblesignal pathsareavailablebetweeninputandoutput asin latticenetwork.
2. Whenthereisinductivecouplingbetweeninputandoutput in additiontoconduction.

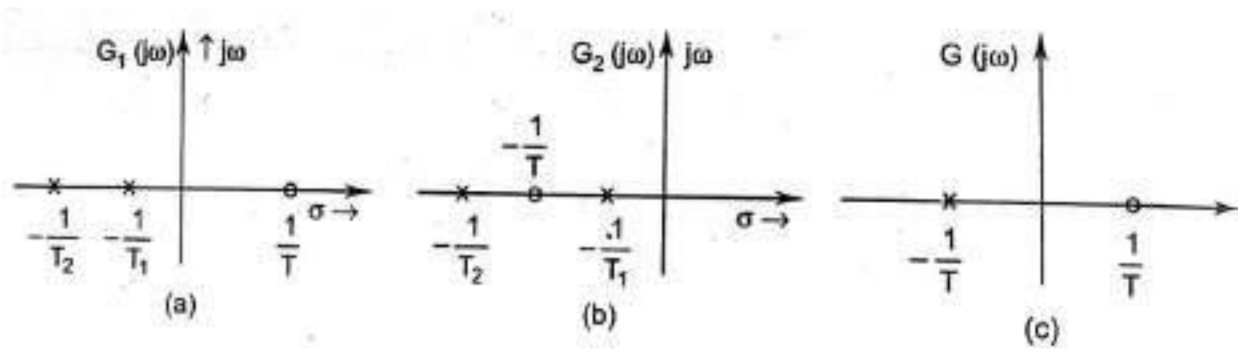


Figure2

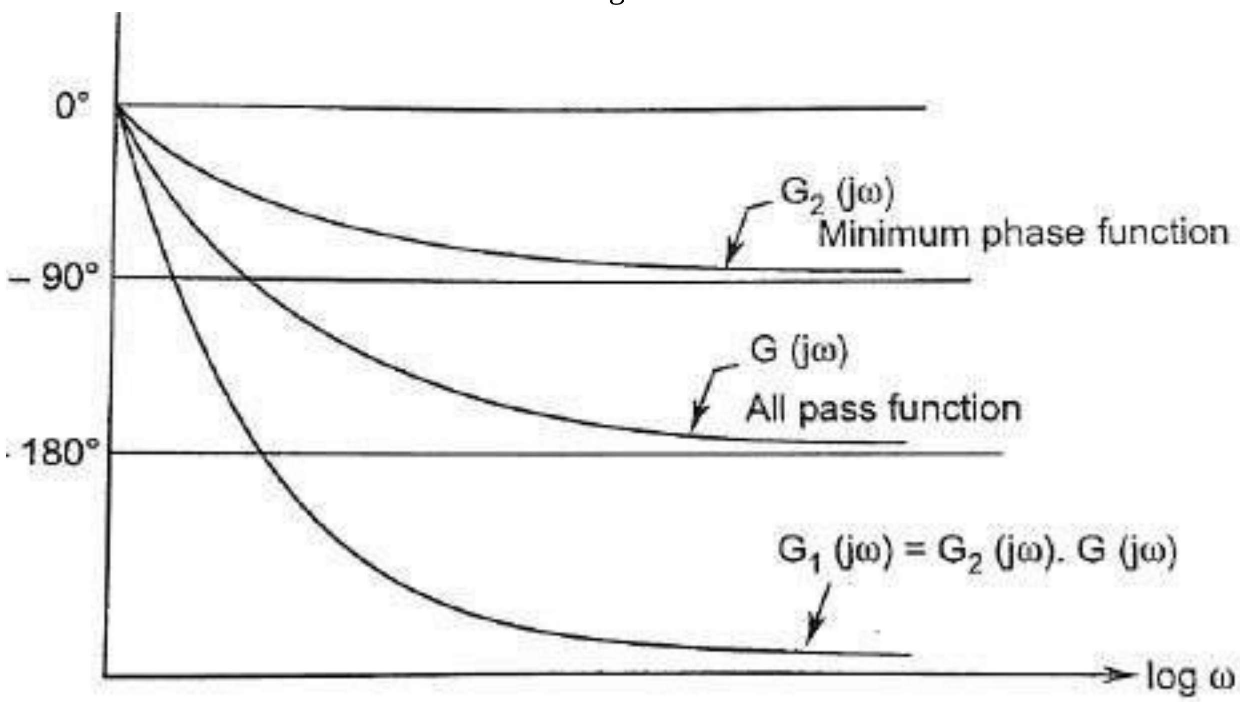


Figure3PhaseVsfrequencygraph

PolarPlots

Polarplotisaplotwhichcanbedrawnbetweenmagnitudeandphase.Itisaplotofmagnitude $|G(j\omega)|$versusphaseangle$\angle G(j\omega)$onpolarco-ordinatesasinputfrequency(ω)isvariedfrom0to∞.Here, themagnitudes are represented bynormal values only.

Thepolarform ofG(jω) is

$$G(j\omega)=|G(j\omega)|\angle G(j\omega)$$

RulesforDrawingPolarPlots

Followtheserulesforplottingthe polarplots.

Step1.Substitute,s=jωintheopenlooptransferfunction.
Step2.Writetheexpressionsformagnitudeand thephaseofG(jω)

Step3.FindthestartingmagnitudeandthephaseofG(jω)bysubstituting ω=0.So,thepolarplotstartswith this magnitude and the phaseangle.
Step4.FindtheendingmagnitudeandthephaseofG(jω)bysubstitutingω=∞.So,thepolarplotendswith thismagnitudeand the phaseangle.
Step5. Check whether thepolarplotintersectstherealaxis,by makingtheimaginary termofG(jω)equal to zeroand find the value(s) ofω.
Step6.Determinetheintersectionofpolarplot withrealaxisandimaginaryaxis,asfollows:

i. Rationalisethefunction G(jω)andseparatethereal andimaginaryparts.
ii. Intersection with imaginary axis: equate the real term of |G(jω)| to zero and find thevalue of frequency (ω) at which the polar plot intersects the imaginary axis. Now putthisvalueofωinto|G(jω)|. Whichgives|G(jω)|atthispoint ofintersection.
iii. Intersection with real axis: equate the imaginary term of |G(jω)| to zero and find thevalueoffrequency(ω)atwhich thepolarplotintersects the realaxis. Nowput thisvalueofωinto|G(jω)|.Whichgives|G(jω)|atthispointofintersection.

Step7.By using this information, plot the points on the complex plane. Make the arrow on theplotforincreasingfrequency from 0 to ∞.

Example1:Considertheopenloop transferfunction ofaclosedloopcontrol system.

$$G(s)=\frac{1}{(1+sT_1)(1+sT_2)}$$

Drawthe polarplot.

Step1−Substitute,s=jωintheopenlooptransfer function.

$$G(j\omega)=\frac{1}{(1+j\omega T_1)(1+j\omega T_2)}$$

Themagnitudeof theopen looptransferfunctionis

$$|G(j\omega)|=\frac{1}{\sqrt{1+(\omega T_1)^2}\sqrt{1+(\omega T_2)^2}}$$

Thephaseangleoftheopen looptransfer function is

$$\angle G(j\omega)=-\tan^{-1}\omega T_1-\tan^{-1}\omega T_2$$

Step2−Thefollowingtableshowsthemagnitudeandthephaseangleoftheopenlooptransferfunctionat ω=0rad/secand ω=∞rad/sec.

Frequency(rad/sec)	Magnitude	Phaseangle(degrees)
0	1	0
∞	0	-180^{o}

So, the polar plot starts at $(1,0^0)$ and ends at $(0,-180^0)$. The first and the second terms within thebracketsindicate themagnitudeand phaseanglerespectively.

Step3−Thispolarplotwillintersectthenegativeimaginaryaxis.Thephaseanglecorresponding to the negative imaginary axis is -90^0 or 270^0. So, by equating the phase angle oftheopenloop transferfunction toeither-90^0or 270^0, wewill get the ωvalueas

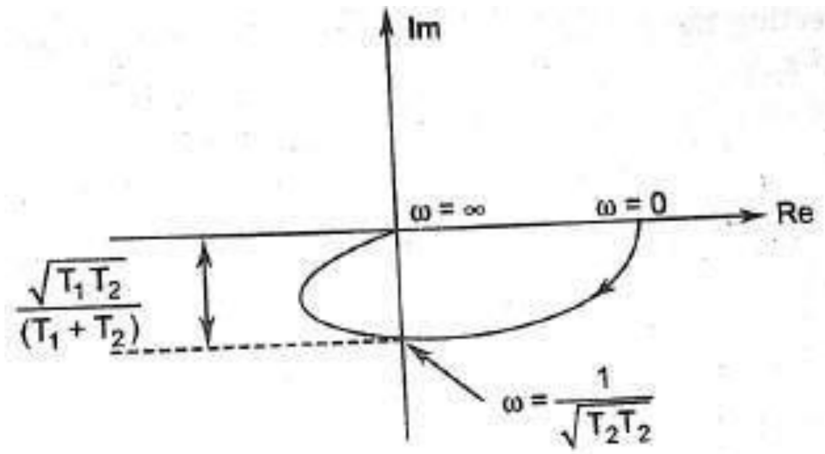
$$\angle G(j\omega)=-\tan^{-1}\omega T_1-\tan^{-1}\omega T_2=-90^0$$

$$\Rightarrow \frac{\omega T_1+\omega T_2}{1-\omega^2 T_1T_2}=\infty \Rightarrow \omega=\frac{1}{\sqrt{T_1T_2}}$$

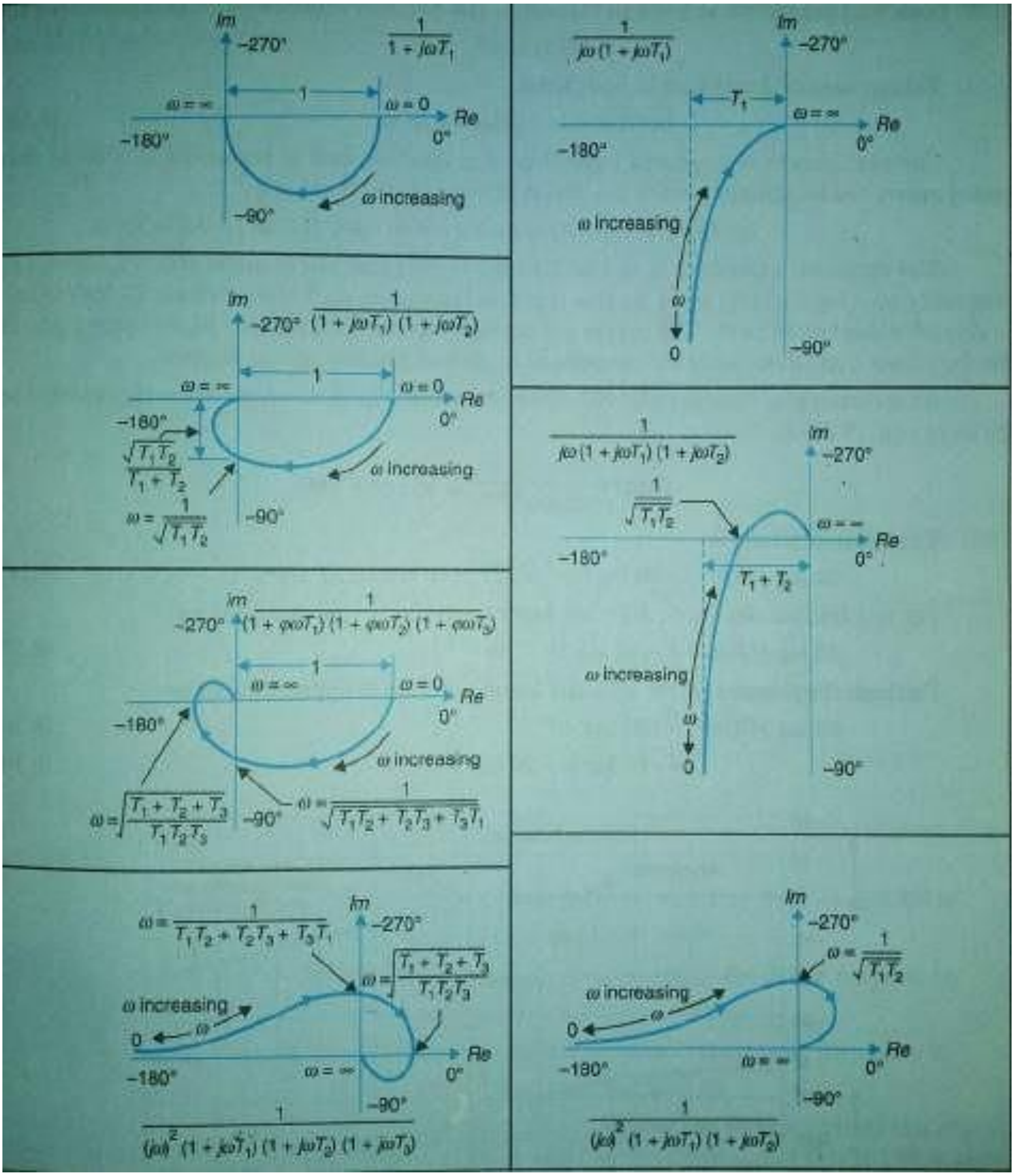
Bysubstitutingω=$\frac{1}{\sqrt{T_1T_2}}$ inthemagnitudeoftheopenlooptransfer function, wewill get

$$|G(j\omega)|=\frac{1}{\sqrt{1+\left(\frac{1}{\sqrt{T_1T_2}}T_1\right)^2}\sqrt{1+\left(\frac{1}{\sqrt{T_1T_2}}T_2\right)^2}}=\frac{\sqrt{T_1T_2}}{T_1+T_2}$$

So,wecan drawthepolar plotwith theaboveinformation onthepolar graphsheet.



Thefollowingtableshowspolat plot fordifferenttypeof control system:



NyquistPlots

Introduction:

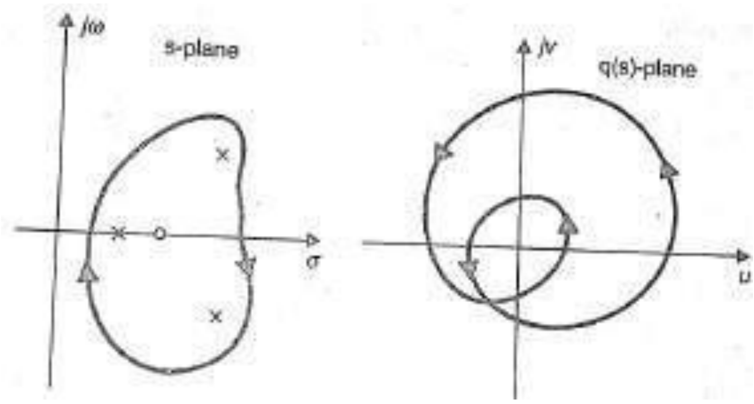
Nyquistplotsarethecontinuationofpolarplotsforfindingthestabilityoftheclosedloopcontrol systems by varying ω from −∞ to ∞. That means, Nyquist plots are used to draw thecompletefrequencyresponseof theopen loop transferfunction.

Principleofargument

The Nyquist stability criterion works on the principle of argument. It states that if there are Ppoles and Z zeros are enclosed by the ‘s’ plane contour, then the corresponding G(s)H(s) planemust encircle theoriginP−Ztimes. So,wecanwritethenumberofencirclements N as,
N=P−Z

- Ifthe‘s’planecontourcontainsonlypoles,thenthedirectionoftheencirclementintheq(s)planewillbeopposite(counterclockwise) tothedirection of‘s’planecontour.
- Ifthe‘s’planecontourcontainsonlyzeros,thenthedirectionoftheencirclementintheq(s)planewill be in thesame(clock wise)direction as thatof ‘s’planecontour.

For example, in case of 1 zero and 3 poles enclosed by the s- plane contour, the net encirclementof the origin by the q(s) plane contour is (3-1) two counter-clockwise revolution as shown infigure below. This relationship between the enclosure of poles and zeros of G(s)H(s) b the s-planecontourandtheencirclementoftheoriginbyG(s)H(s)contouriscommonlyknownasprincipleofargument.



Nyquiststabilitycriterion

Thecharacteristicsequationofasystemisq(s)=1
+G(s)H(s)
Thestandardpole zero form oftheOLTFG(s)H(s) is

$$G(s)H(s)=K\frac{(s+z_1)(s+z_2)-------(s+z_m)}{(s+P_1)(s+P_2)--------(s+P_n)} \quad (1)$$

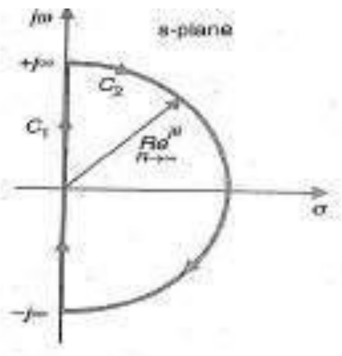
$$\therefore q(s) = 1 + K\frac{(s+z_1)(s+z_2)\text{--------}(s+z_m)}{(s+P_1)(s+P_2)\text{---------}(s+P_n)}$$

$$= \frac{(s+P_1)(s+P_2)\text{---------}(s+P_n) + K(s+z_1)(s+z_2)\text{--------}(s+z_m)}{(s+P_1)(s+P_2)\text{---------}(s+P_n)}$$

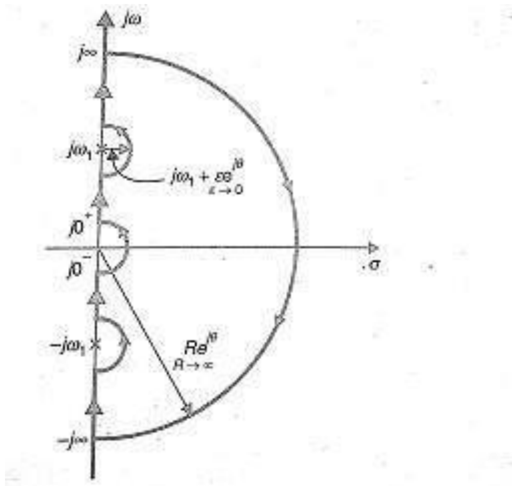
$$= \frac{(s+z_1^F)(s+z_2^F)\text{---------}(s+z_n^F)}{(s+P_1)(s+P_2)\text{---------}(s+P_n)} \qquad (2)$$

From the above equation it is seen that the zeros of q(s) are the root of the characteristics equationand the poles q(s) are same as the poles of open loop system. For the system to be stable, the rootsof the characteristics equation and hence the zeros of q(s) must lie in the left half s-plane. It isimportant to note that even if some of the open-loop poles lie in the right half s-plane all thezeros of q(s) i.e, the closed-loop poles may lie in the left half s-plane. It means that an open-loopunstablesystem maylead to a closed-loop stablesystem.

In order to investigate the presence of any zeroof q(s) in the right half of s-plane, acontour to be chosenwhich completely encloses the right half of s-plane calledasNyquistcontour. It is directed clockwise and consist of an infinite line segment $C_1$and an arc C_2 ofinfiniteredius.



The Nyquist contour.



Indented Nyquist contour for jω-axis open-loop poles.

As the Nyquist contour encloses all the right half s-plane poles and zeros of q(s), let there are ‘z’zeros and ‘P’ poles in the right half of s-plane. As s moves along the nyquist contour in the s-plane, a closed contour Γ_q is traversed in q(s) plane which encloses the origin N (=P-Z) times inanticlockwisedirection.

Forthesystem to bestable, thereshould benozeros of q(s) in theright halfof s-planei.e,

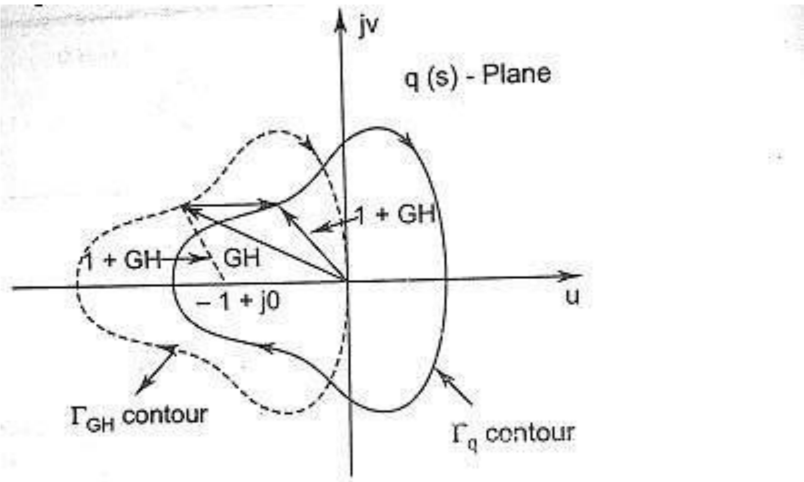
$$Z = 0 \text{ So } N = P$$

The above equation implies that for a close loop system to be stable, the number of counter-clockwise encirclement of the origin of the q(s) plane by the contour $\Gamma_q$should be equal thenumber of the right half s-plane poles of q(s) which are also the poles of open-loop transferfunctionG(s)H(s).

Theopen-looptransferfunctioncanbewrittenas

$$G(s)H(s) = q(s) - 1 = [1 + G(s)H(s)] - 1 \qquad (3)$$

Therefore the contour Γ_{GH} of G(s)H(s) corresponding to the nyquist contour in the s-plane is thesame as contour Γ_q of q(s) (=1+ G(s)H(s)) drawnfromthe point(-1+j0). Thus the encirclementof the origin by the contour Γ_q of q(s) is equivalent to the encirclement of the point (-1+j0) by thecontour$\Gamma_{GH}$of G(s)H(s)as shown below.



Contour Γ_{GH} of G(s) H (s) corresponding to Nyquist contour.

Statementofnyquist stabilitycriterion:

1. If the contour Γ_{GH} corresponding to the Nyquist contour in the s-plane encircles the point (-1+j0) in the counter-clockwise direction as many times as the number of right half s-plane poleofG(s)H(s), theclose loop system is stable.
2. Theclosedloopsystem is stableifthecontourΓ_{GH}doesnotencircles thepoint (-1+j0).

MappingofNyquistcontourintothecontourΓ_{GH}ofG(s)H(s):

1. Forimaginaryaxis:Put s=jωinG(s)H(s)wheresvariesfrom–j∞to+j∞.

2. Forinfinitesemicircle:puts= $Re^{j\theta}$whereR→∞andθvariesfrom+90°to-90°.

3. For presenceof poleatorigin:puts=$\in e^{j\theta}$where∈→∞andθvariesfrom-90°to+90°.

4. Forpresenceofpoleatimaginaryaxis:puts=$j\omega_1+\in e^{j\theta}$where∈→∞andθvariesfrom-90°to +90°.

HencethecompletecontourΓ_{GH}isthepolarplotofG(jω)H(jω)withvariesfromω -∞to +∞.

NyquiststabilitycriterionappliedtoinversePolarplot:

It is more convenient to work with inverse function 1/ G(jω)H(jω) rather than the direct functionG(jω)H(jω). Here we will see that the Nyquist stability criterion for direct polar plot can beextendedforuseto inverse polarplot after minormodification.

Letusconsideraopen-looptransferfunction:

$$G(s)H(s)=K\frac{(s+z_1)(s+z_2)--------(s+z_m)}{(s+P_1)(s+P_2)---------(s+P_n)} \qquad (4)$$

Forthesystemtobestablenoneoftherootsofthecharacteristicsequationshouldlieintherighthalfs-planeor onthejω-axis. The characteristicsequation is

$$q(s)=1+G(s)H(s)=\frac{(s+z_1^F)(s+z_2^F)---------(s+z_n^F)}{(s+P_1)(s+P_2)--------(s+P_n)} \tag{5}$$

Dividingequation 5by4,weget

$$q'(s)=\frac{1}{G(s)H(s)}+1=\frac{(s+z_1^F)(s+z_2^F)---------(s+z_n^F)}{(s+z_1)(s+z_2)-------(s+z_m)} \tag{6}$$

Fromequation5and6itisseenthatthezerosofq'(s)issameastheq(s),whicharetherootsofthecharacteristic sequation.Itisfurthernoticedthatthepolesofq(s)aresameasthepolesof G(s)H(s),whilethepolesofq'(s)aresameasthepolesof $\frac{1}{G(s)H(s)}$ orthezerosofG(s)H(s).

Itcanbeconcludedthatif $\frac{1}{G(s)H(s)}$ hasPrighthalfs-planepolesandthecharacteristics equationhasZrighthalfs-planezeros,thelocusof $\frac{1}{G(s)H(s)}$ encirclethepoint(-1+j0)Ntimesin counter-clockwisedirectionwhereN=P-Z.

Sinceforsystemstabilitynozerosofthecharacteristicsequationlocateonrighthalfs-planei.e ,Z=0, theNyquiststabilitycriterion for inversepolarplots canbestated below:

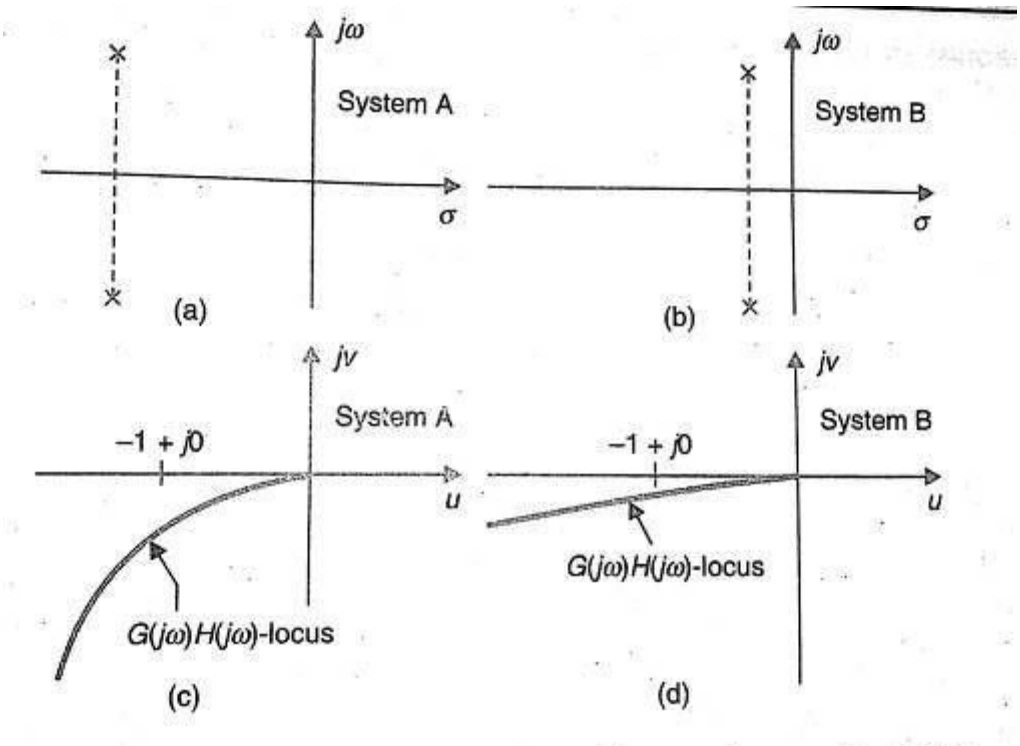
"IttheNyquistplotof $\frac{1}{G(s)H(s)}$ correspondingtotheNyquistcontourinthes-plane,encircles counter-clockwisethe point(-1+j0) asmanytimesasarethenumberofrighthalfs-planepole of $\frac{1}{G(s)H(s)}$,theclosed-loop systemisstable."

Inspecialcasewhere $\frac{1}{G(s)H(s)}$ has nopoleintheright halfs-plane,thecloseloop systemis stableprovidedthe netencirclement of (-1+j0)point bytheNyquistplot of $\frac{1}{G(s)H(s)}$ iszero.

AssessmentofrelativestabilityusingNyquistcriterion:

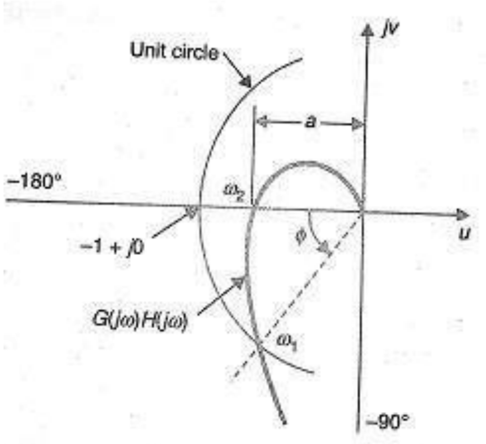
The measure of relative stability of a closed-loop systems which are open-loop stable canbe analysed through the study of Nyquist plots. The stability of such system can be determined bypolar plot of G(s)H(s). It can be imagined that as the polar plot gets closer to (-1+j0) point, thesystemtends towards instability.

Consider two different systems whose closed loop poles are shown on the s-plane in figurea and b respectively. It is seen that system A is more stable than system B because its closed-looppoles are located comparatively away to the left from jω-axis. The open-loop frequency response(polar) plots for system A and B are shown in figure 'c' and 'd', respectively. The comparison ofthe closed-loop pole location of these twosystem with theircorresponding polar plot shows thatas a polar plot moves closer to (-1+j0) point, the system closed-loop poles move closer to the jω-axisand hencethe system becomesrelativelyless stable andviceversa.



Correlation between the closed-loop s-plane root locations and open-loop frequency response curves.

The figure as given below shows a G(jω)H(jω) locus which crosses the negative real axis at afrequency $\omega=\omega_2$ with an intercept of a. Let a unit circle centred at origin (passes through point -1+j0) intersect the G(jω)H(jω) locus at a frequency $\omega=\omega_1$ and let the phasor$G(j\omega_1)H(j\omega_1)$ makesan angle of ϕ with the negative real axis measured positively in counter-clockwise direction. It isobservedthatasG(jω)H(jω)locusapproaches(-1+j0)point, therelativestabilityreduces.



A typical G(jω)H(jω)-locus.

ConstantMagnitudeLociorConstantMCircle

Theclosedlooptransferfunctionofaunityfeedbacksystem is givenby

$$T(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

or
$$T(j\omega) = \frac{C(j\omega)}{R(j\omega)} = \frac{G(j\omega)}{1+G(j\omega)}$$

Let
$$G(j\omega) = x + jy$$

$\therefore$
$$T(j\omega) = \frac{(x+jy)}{1+(x+jy)} = \frac{x+jy}{1+x+jy} \qquad (1)$$

Let magnitude of T $(j\omega)$ is M, we can write

$$M = \frac{|x+jy|}{|1+x+jy|} = \frac{\sqrt{x^2+y^2}}{\sqrt{(1+x)^2+y^2}} \qquad (2)$$

Squaring both sides, we get

$$M^2 = \frac{x^2+y^2}{(1+x)^2+y^2} \qquad (3)$$

or
$$M^2[(1+x)^2+y^2] = x^2+y^2$$

or
$$M^2[1+x^2+2x+y^2] = x^2+y^2$$

or
$$M^2[x^2+y^2+2x+1] - x^2 - y^2 = 0$$

or
$$x^2(M^2-1) + 2xM^2 + y^2(M^2-1) + M^2 = 0$$

or
$$x^2 + \frac{2x}{(M^2-1)} . M^2 + y^2 + \frac{M^2}{(M^2-1)} = 0$$

or
$$x^2 - \frac{2x}{(1-M^2)} . M^2 + y^2 - \frac{M^2}{(1-M^2)} = 0$$

or $$x^2 - \frac{2xM^2}{(1-M^2)} + y^2 = \frac{M^2}{(1-M^2)}$$

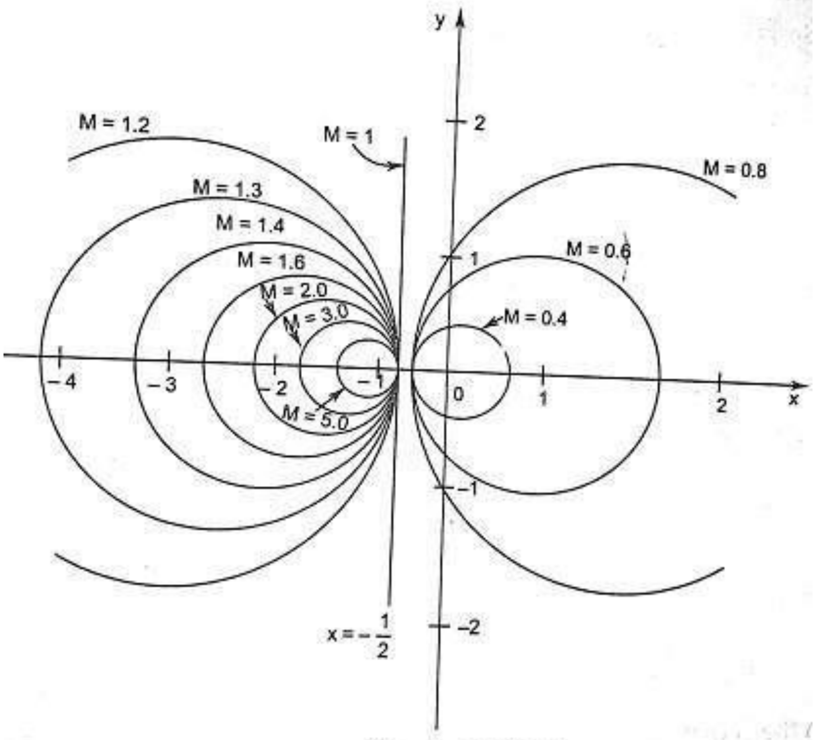
or $$x^2 - \frac{2xM^2}{(1-M^2)} + \left[\frac{M^2}{(1-M^2)}\right]^2 + y^2 = \frac{M^2}{(1-M^2)} + \left[\frac{M^2}{(1-M^2)}\right]^2$$

or $$\left[x - \frac{M^2}{(1-M^2)}\right]^2 + y^2 = \frac{M^2}{(1-M^2)^2} \qquad (4)$$

Equation2represents the equationofacirclewithcentreat[$\frac{M^2}{(1-M^2)}$,0]havingradiusof $\frac{M}{(1-M^2)}$

IfM=1,thenEquation3becomes$(1+x)^2+y^2=x^2+y^2$ or$x=-\frac{1}{2}$ (5)

Itis aequation forstraightlineparallel tothey-axisand passingthrough$(-\frac{1}{2},0)$ intheG(jω) plane. For each value of M (except M=1) we get a circle. These circles are known as ConstantMagnitude Locior ConstantM Circle.



Constant M-circles.

ConstantPhaseLociorConstantNCircle

Fromequation1

$$T(j\omega) = \frac{C(j\omega)}{R(j\omega)} = \frac{(x+jy)}{(1+x)+jy}$$

Phase angle of T $(j\omega)$ is given by

$$\angle T(j\omega) = \angle\left[\frac{C(j\omega)}{R(j\omega)}\right] = \angle\left[\frac{(x+jy)}{(1+x)+jy}\right]$$

$$\Rightarrow \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right) - \tan^{-1}\left(\frac{y}{1+x}\right) \tag{6}$$

$$\Rightarrow \qquad \tan\theta = \frac{\dfrac{y}{x} - \dfrac{y}{1+x}}{1 + \dfrac{y^2}{x(1+x)}}$$

$$= \frac{y}{x^2 + x + y^2}$$

$$\text{Let} \qquad \tan\theta = N \tag{7}$$

$$\therefore \qquad N = \frac{y}{x^2 + x + y^2} \tag{8}$$

$$\text{or} \qquad N(x^2 + x + y^2) = y$$

$$\text{or} \qquad x^2 + x + y^2 - \frac{y}{N} = 0$$

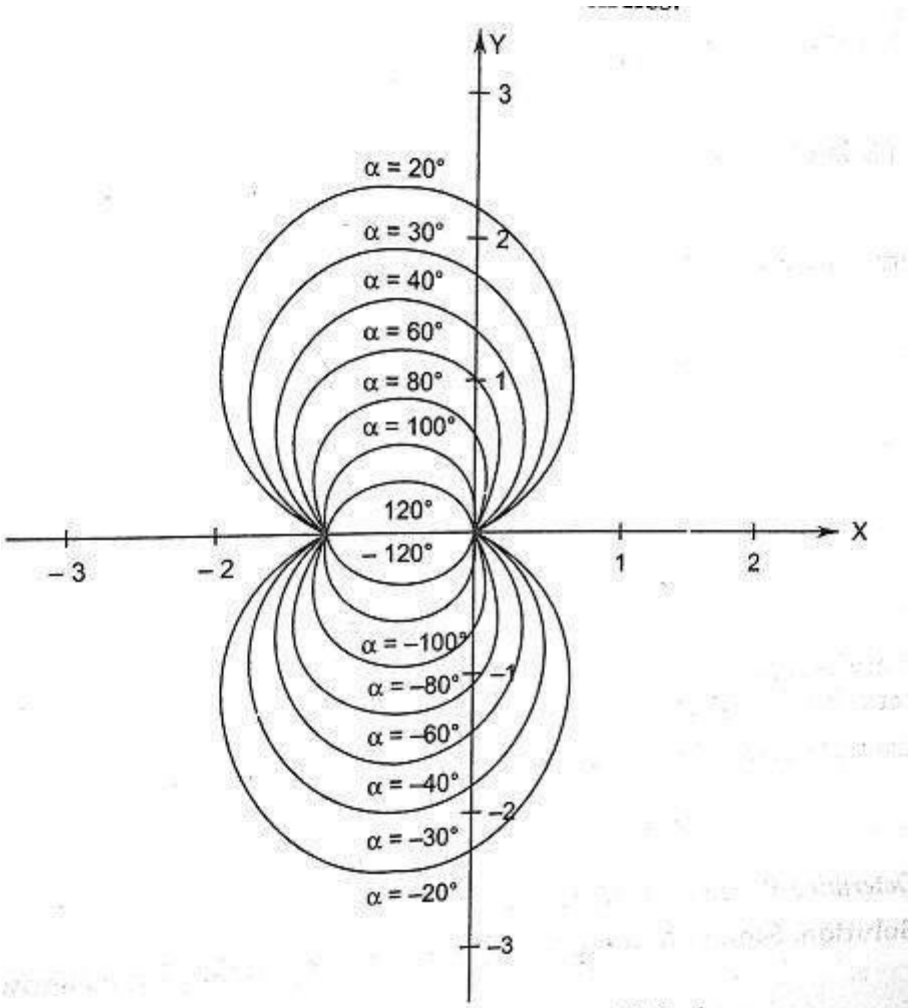
Add $\left(\frac{1}{4} + \frac{1}{4N^2}\right)$ to both sides, we get (9)

$$x^2 + x + y^2 - \frac{y}{N} + \frac{1}{4} + \frac{1}{(4N^2)} = \frac{1}{4} + \frac{1}{4N^2}$$

$$\text{or} \qquad \left(x + \frac{1}{2}\right)^2 + \left(y - \frac{1}{2N}\right)^2 = \frac{1}{4} + \frac{1}{4N^2} \tag{10}$$

Equation10representstheequationofcirclewithitcentreat$\left(-\frac{1}{2}, \frac{1}{2N}\right)$withradius$\sqrt{\left(\frac{1}{4} + \frac{1}{4N^2}\right)}$

For different values of N i.e, phase angle θ, equation 10 represents the family of the circles. For aparticular circle, the value of N i.e, phase angle θ remain constant on it. Therefore these circle areknownas constant phaseloci orN circles.



A family of constant N circles.

<u>NicholasChart</u>

Constant magnitude loci that are M-circles and constant phase angle loci that are N-circles are thefundamental components in designing the Nichols chart. The constant M and constant N circles inG (jω) plane can be used for the analysis and design of control systems. However the constant Mand constant N circles in gain phase plane are prepared for system design and analysis as theseplots supply information with fewer manipulations. Gain phase plane is the graph having gain indecibel along the ordinate (vertical axis) and phase angle along the abscissa (horizontal axis). TheM and N circles of G (jω) in the gain phase plane are transformed into M and N contours inrectangular co-ordinates.A point on the constant M loci in G (jω) plane is transferredto gainphaseplanebydrawingthevectordirectedfromtheoriginofG(jω)planetoaparticularpointon M circle and then measuring the length in db and angle in degree. Thecritical point in G (jω), plane corresponds to the point of zero decibel and -180° in the gain phaseplane.Plot ofMand N circles ingain phaseplaneisknown as Nicholschart/plot.

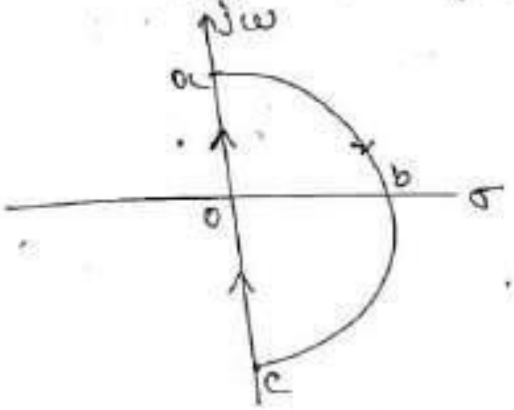
The Nichols chart is named after the American engineer N.B Nichols who formulated this plot.CompensatorscanbedesignedusingNicholsplot.Nicholsplottechniqueishoweveralsousedin designing of dc motor. This is used in signal processing and control design. Nyquistplot incomplex plane shows how phase of transfer function and frequency variation of magnitude arerelated. We can find out the gain and phase for a given frequency. Angle of positive real axisdeterminesthephaseanddistancefromorigin ofcomplexplanedeterminesthegain.

Example:1 - Sketch Nyquist plot for a given OLTF.

$$G(s)H(s) = \frac{K}{(T_1 s+1)(T_2 s+1)}$$

Solution

step1: First draw nyquist path for the given OLTF.



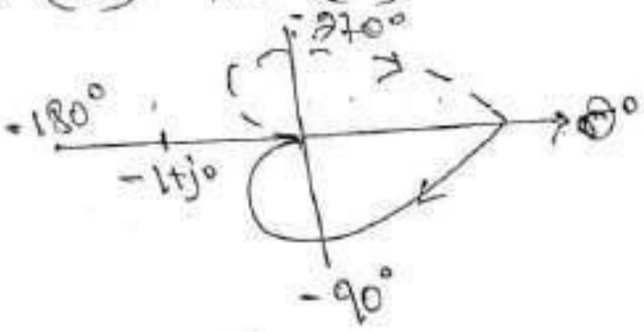
step2: For path oa put $s=j\omega$ where ω varies from 0 to ∞ & plot polar plot.

$$G(j\omega)H(j\omega) = K / (T_1 j\omega+1)(T_2 j\omega+1)$$

$$M = |G(j\omega)H(j\omega)| = K / \left[\sqrt{\{(T_1\omega)^2+1\}\{(T_2\omega)^2+1\}}\right]$$

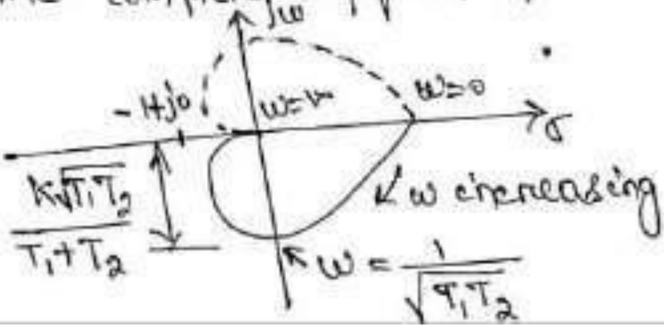
$$\phi = \angle G(j\omega)H(j\omega) = -\tan^{-1}(\omega T_1) - \tan^{-1}(\omega T_2)$$

ω	M	ϕ
0	K	0°
∞	0	-180°



Step 3: Path co is the mirror image of path oa shown by a doted line in the figure as given below,

Step 4: The complete nyquist plot is;

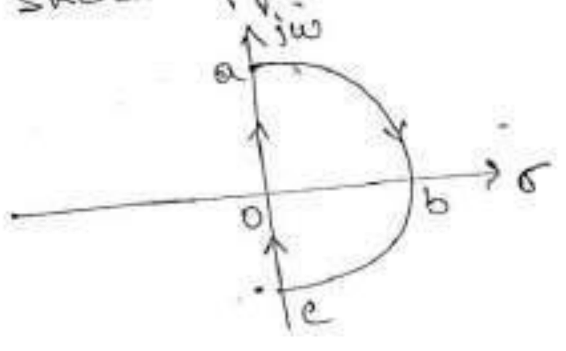


Step 5: It is seen that the plot of $G(j\omega)$ does not encircle the point $(-1+j0)$ i.e $N=0$, therefore the system is stable.

Example 2: Sketch Nyquist plot for the given OLTF.

$$G(s)H(s) = \frac{s+2}{(s+1)(s-1)}$$

Sol^n Step 1: Sketch Nyquist path for the OLTF.



Step 2: For path oa put $s=j\omega$ & sketch polar plot.

$$G(j\omega)H(j\omega) = \frac{2+j\omega}{(1+j\omega)(-1+j\omega)}$$

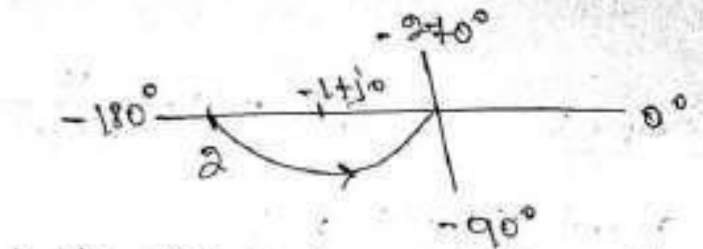
$$M = |G(j\omega)H(j\omega)| = \frac{\sqrt{4+\omega^2}}{\sqrt{1+\omega^2}\sqrt{1+\omega^2}}$$

$$\phi = \angle G(j\omega)H(j\omega) = \tan^{-1}\frac{\omega}{2} - \tan^{-1}\frac{\omega}{1} - \tan^{-1}(-\omega)$$

$$= \tan^{-1}\frac{\omega}{2} - \tan^{-1}\omega - (180 - \tan^{-1}\omega)$$

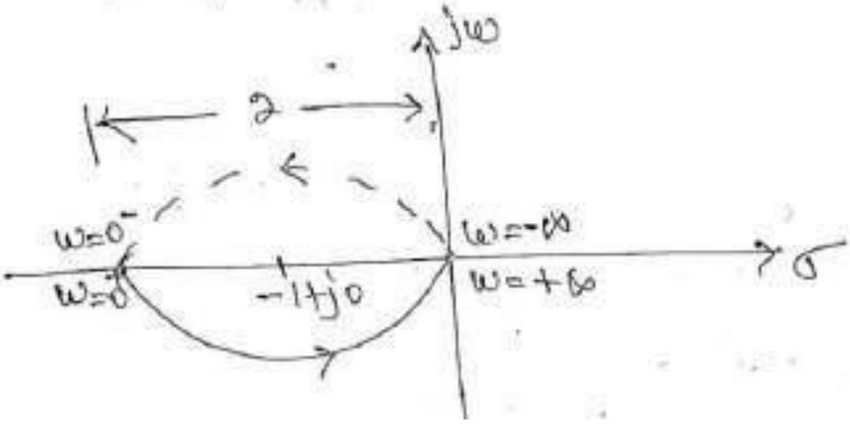
ω	M	ϕ
0	2	-180°
∞	0	-90°

Polar plot is



Step 3: Path CO is the mirror image of the path oa, as shown below:

Step 4: The complete contour for $F_1(j\omega)$ is as given below,



Step 5: Check for stability

From the OLTF no. of pole present in the right half of s-plane $P = 1$.

From Nyquist plot no of counter clockwise encirclement $N = 1$.

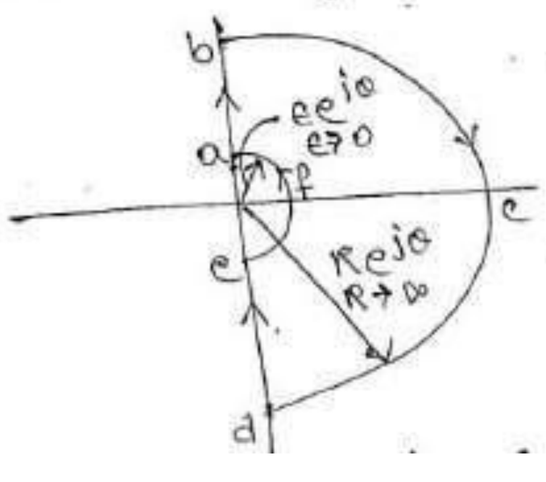
$\therefore N = P - Z \Rightarrow 1 = 1 - Z \Rightarrow Z = 0$

which states that there are no zeros of $1+G(s)H(s)$ in the right half s-plane and hence the closed-loop system is stable.

Example 3 $\quad G(s)H(s) = \frac{K}{s(Ts+1)}$

Sketch Nyquist plot?

Soln Step 1: As the given OLTF has pole at origin the Nyquist contour is as given below:



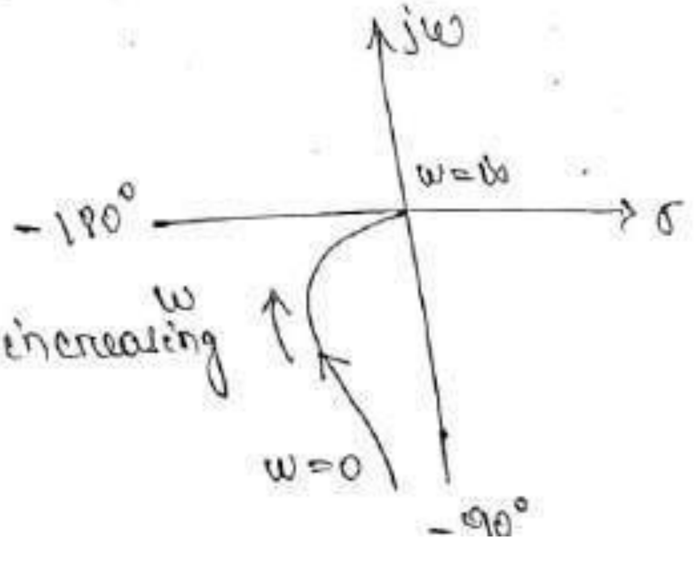
Step2: For path ab put $s=j\omega$ & where ω varies from 0 to $+\infty$) and sketch polar plot.

$$G(j\omega)H(j\omega) = \frac{K}{j\omega(j\omega T+1)}$$

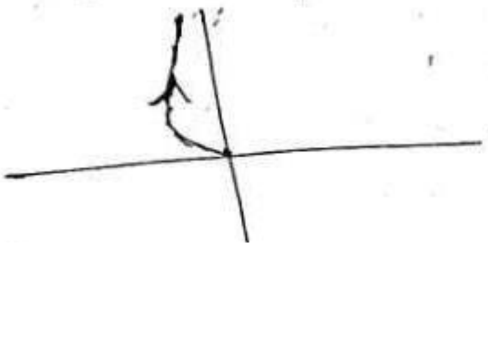
$$M = |G(j\omega)H(j\omega)| = \frac{K}{\omega\sqrt{(\omega T)^2+1}}$$

$$\phi = \angle G(j\omega)H(j\omega) = -90^\circ - \tan^{-1}(\omega T)$$

$\omega=0$	$M=\infty$	$\phi=-90^\circ$
$\omega=\infty$	$M=0$	$\phi=-180^\circ$



Step3: For path de, it is the mirror image of path ab where ω varies from $-\infty$ to 0.



Step 4: For path "bcd" put $s = Re^{j\theta}$ where $R \to \infty$ and θ varies from $+90°$ through $0°$ to $-90°$.

$$G(j\omega)H(j\omega)\Big|_{\substack{s=Re^{j\theta} \\ R\to\infty}} = \lim_{R\to\infty} \frac{K}{Re^{j\theta}(TRe^{j\theta}+1)}$$

$$= \lim_{R\to\infty} \frac{K}{TR^2 e^{j2\theta}} = 0e^{-j2\theta}$$

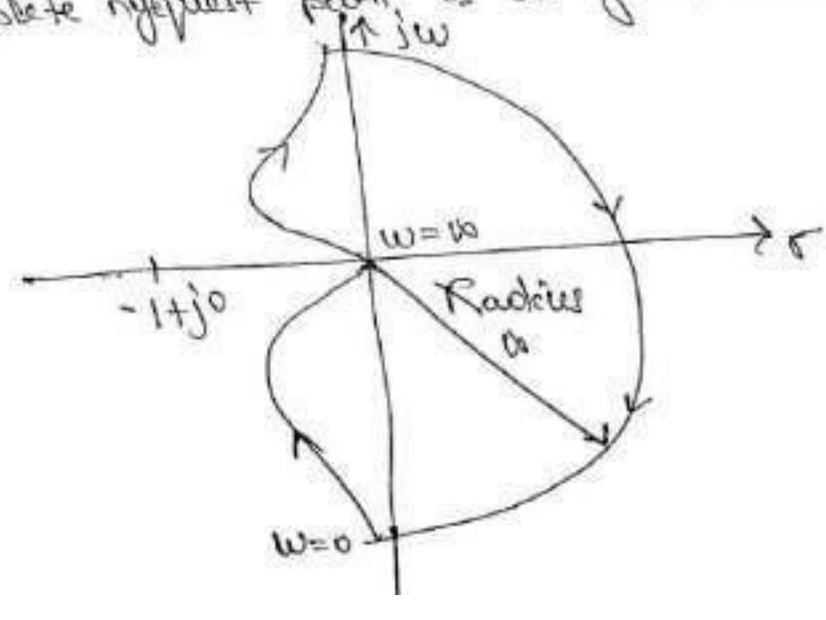
$\therefore$ G(s) H(s) locus turns at the origin with zero radius from $-180°$ through $0°$ to $+180°$.

Step 5: For path "efa" put $s = \epsilon e^{j\theta}$ where $\epsilon \to 0$ and θ varies from $-90°$ through $0°$ to $+90°$.

$$G(j\omega)H(j\omega)\Big|_{\substack{s=\epsilon e^{j\theta} \\ \epsilon\to 0}} = \lim_{\epsilon\to 0} \frac{K}{\epsilon e^{j\theta}(T\epsilon e^{j\theta}+1)}$$

$$= \lim_{\epsilon\to 0} \frac{K}{\epsilon e^{j\theta}} = \infty e^{-j\theta}$$

Thus G(s)H(s) locus turns at the origin with ∞ radius from $+90°$ through $0°$ to $-90°$. Hence the complete nyquist path is as given below:



Step-6:Checkforstability:

AsP=0;N=0; HenceZ=0,Systemisstable.

Therearesome advantagesofNichols'plotincontrolsystem engineering.

- Gainandphasemarginscanbedeterminedeasilyandalsographically.
- Closedloopfrequencyresponseisobtainedfrom openloopfrequencyresponse.
- Thegainof thesystem canbeadjustedto suitable values.
- Nicholschartprovidesfrequencydomainspecifications.

The related Nyquist plot in the complex plane shows how the phase of the transfer function andfrequency variation of magnitude are related. We can find out the gain and phase for a givenfrequency.

The angle of the positive real axis determines the phase and the distance from the origin of thecomplexplanedetermines the gain.